

The Good, the Bad and the Ugly

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ABSTRACT:

We build and linearize a new analytical general equilibrium model with “good” and “bad” forms of waste disposal, when “ugly” illegal disposal cannot be taxed or regulated. This model can apply to any pollution problem in the air, water, or land. We find closed-form solutions for changes in each outcome from a small change in tax, and we use those expressions in a welfare function to find closed-form solutions for levels of the two second-best optimal tax rates (which depend on all damages, elasticities, and share parameters). The tax on bad disposal might be less than the tax on good disposal, or even less than zero, if bad disposal is the best substitute for ugly disposal. We calibrate the model for numerical illustrations and show how partial equilibrium elasticities and parameters combine with resource constraints and revenue requirements to determine very different general equilibrium elasticities for effects on each outcome. These general equilibrium outcomes are what matter to set the two optimal tax rates.

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Any type of pollution or waste involves production or disposal options that might include a “good” category of choices that are relatively clean: electric cars, methane capture, or waste recycling. Another category is “bad” options that are dirtier but still legal: gasoline cars, flaring methane, or waste into legal landfills. Unfortunately, however, firms or households can also make cheaper “ugly” choices: illegal emissions into air, water, or land. In Pigou (1920), a welfare-maximizing planner imposes a first-best optimal (FBO) tax on each activity at a rate equal to its marginal external damage (MED). But enforcing rules on illegal activities are often too costly or impossible. Even if illegal dumping is a small fraction of total waste, its high MED can imply large total social costs. As a result, much empirical research in economics has measured how these production or disposal choices respond to taxes or regulation.

This empirical literature is wide and deep. Table 1 lists eight diverse examples including industrial and household waste disposal options and choices about production and emissions into air or water. In one example, vehicles can involve zero emissions, gasoline combustion, or ugly choices like driving with a failed emissions test. The eight listed papers measure behavior and discuss policy. But none has a theoretical framework with formal statements about conditions under which each taxable choice would optimally be taxed more or less than others.

This paper builds a new analytical general equilibrium (GE) model with three waste options (good, bad, and ugly). We also solve two special cases. We use each model to derive second-best optimal (SBO) tax or subsidy rates on the taxable good or bad waste categories. Our models are general enough that production choices about emissions are equivalent to household choices about final disposal. Any disposal option could be untaxable, with the same results, but we focus on the case where the dirtiest option is the one that cannot be taxed or regulated. We also show how GE elasticities differ substantially from partial equilibrium (PE) elasticities.

In any model, subsidies to legal waste disposal would not be surprising, as virtually all eight papers listed in Table 1 make that suggestion (*e.g.*, subsidize low-polluting methods to avoid worse outcomes).¹ But subsidizing all cleaner options is too easy a solution. Even a benevolent social planner must make difficult choices in the face of resource and budget constraints. For this reason, we also show implications of a revenue constraint, requiring a tax on

¹ Fullerton and Kinnaman (1995) study garbage, recycling, and illicit dumping, and they derive a FBO deposit-refund system. Thus, a deposit-refund system might circumvent the need for SBO tax rates. For a case that necessitates the second-best policies in this paper, it must be that the FBO deposit-refund system is not available. For example, politicians may like a subsidy to recycling but abhor a tax (deposit) on output. Or perhaps coordination is too difficult when a regional level of government controls output taxes and local government controls recycling policy. In any case, that paper does not consider second-best optimal tax rates, as done here.

at least one option. Our closed-form solutions show how SBO tax rates depend on elasticity and share parameters. In many cases, the good choice gets a subsidy and the bad one gets taxed. But we also show wide sets of parameter configurations where the badly polluting option is optimally subsidized, and the revenue requirement means that the clean option must pay tax.

Table 1: A Few Examples from Empirical or Numerical Economics Literature

Example	Production or Disposal Methods	Result suggesting a subsidy to a dirty (<i>i.e.</i> , “bad”) form of disposal
Toxic chemical waste	Good: recycling Bad: legal disposal Ugly: dumping	Sullivan (1987): Optimal subsidy for legal disposal, 34.9% of price, “is more efficient than the optimum enforcement program” (p.65).
Household solid waste	Good: recycling Bad: garbage Ugly: dumping	Fullerton and Kinnaman (1996): An 80-cent price per bag reduces garbage but increases dumping by a third of that amount.
Informal sector brick oven fuels (Mexico)	Good: renewables Bad: propane Ugly: used tires	Blackman and Bannister (1998): Encouraging propane can avoid “cheap, highly polluting fuels such as used tires and scrap wood” (p.1).
Used motor oil	Good: re-refining Bad: legal disposal Ugly: dumping	Sigman (1998): Illegal dumping of used motor oil rises in response to higher costs of legal waste disposal or higher costs of re-refining for reuse.
Industrial waste (Japan)	Good: recycling Bad: legal disposal Ugly: dumping	Ichinose and Yamamoto (2011): Funding the provision of additional legal waste treatment facilities would reduce illegal dumping.
Shale gas wells affect surface water	Good: reuse, recycling Bad: treated discharge Ugly: direct discharge	Olmstead <i>et al</i> (2013): “Direct discharge of fluids from well sites is illegal [but could be collected for] reuse, recycling, treatment, and disposal.” (p. 4962)
Methane from oil and gas supply chain	Good: capture Bad: flaring Ugly: venting/leaking	Agerton <i>et al</i> (2023): Capture can produce energy, while “leaked volumes often are either not known or not reported” (p.267).
Vehicle emissions	Good: electric vehicle Bad: tested emissions Ugly: non-compliant	Bradt and Dorsey (2026): A new method to estimate the fraction of cars being driven with fraudulent emission test results.

Part of this problem was captured 70 years ago in a simple model with no externalities. In Corlett and Hague (CH, 1953), uniform tax rates are optimal if applied to all goods in utility, but leisure is untaxable. As an indirect tax on leisure, the SBO tax is lower on the better substitute for leisure and higher on a lesser substitute or complement to leisure. Many later papers study second-best tax policy with untaxed leisure, but not an untaxed choice other than leisure.²

² Except Sallee (2025), discussed below. Google Scholar’s 951 citations to CH (1953) study second-best commodity taxation when leisure is not taxable. Diamond (1973) and Green and Sheshinski (1976) include externalities and

Here, we ignore leisure, assume fixed labor, and study an untaxable waste. The intuition is similar: put a lower tax or subsidy on an option that substitutes most easily for ugly disposal options. Optimal enforcement rules for audit and penalty rates are already well studied (see the review by Slemrod, 2019). Instead, we study the case where a tax on ugly waste is simply not feasible. The SBO tax on bad waste might then be positive or negative, depending not only on the MED of each disposal method but also on the cross-price elasticities.

Also, early optimal tax literature obtains second-best results by interpreting the social planner's first-order conditions; they find closed-form solutions only in special cases (see *e.g.*, Atkinson and Stiglitz, 1976). Here, our models are used not just to compare FOC's but to find a closed-form solution for each SBO tax rate. For tractability, we ignore uncertainty, dynamics, market power, and distributional issues. We assume that many identical households rationally maximize utility from two commodities, where only one creates waste with three disposal options.³ We differentiate all production functions and resource constraints to find 19 linear equations that can be solved to find equilibrium effects of a small change in a tax or subsidy rate on any of the 19 unknown outcomes (changes in goods prices, disposal prices, and all quantities).

Many papers use analytical GE models to find changes in outcomes from a change in tax, but we use those outcome changes to find solutions for SBO tax rate *levels*. We differentiate the utility function to show the change in consumer welfare, given initial tax rates and parameters, and we insert our solutions for the GE change in each outcome. We can then solve for the levels of initial tax rates from which a tax change does not raise welfare. In this way, we use GE formulas for changes in outcomes to solve for optimal tax rate levels.

The model of Sallee (2025) also has untaxed pollution. He focusses on subsidies to clean goods when politicians choose not to tax coal or gasoline, while we focus on illegal emissions that cannot be taxed. Below, we reduce our GE model to his simpler model to show how our results depend on key features not in Sallee's model (*e.g.*, economy-wide resource and revenue constraints). Sallee's model nicely demonstrates his point that clean subsidies require much more information than pollution taxes, especially all cross-price effects, an issue relevant to our

heterogeneous demands, rather than pollution *per se*. Optimal tax literature is reviewed in Sandmo (1976) and Auerbach and Hines (2002). Others discuss taxing a polluting good when leisure is untaxable (*e.g.*, Bovenberg and Goulder, 1996; West and Williams, 2007). No paper that cites CH (1953) replace leisure as the untaxed good.

³ We could have a vector of produced goods in utility, or a vector of more than three disposal options, but the basic policy problem for production or disposal options of any one commodity is basically the same as for our one good that has three options. Our three options require a 3x3 matrix of own-price and cross-price elasticities, but it has all the same intuition as a more complicated model with four or more methods of disposal and a 4x4 or larger matrix.

analysis. But his paper is not about the problem in prior second-best tax literature of designing tax rates to collect a fixed revenue by use only of distorting taxes. Instead, in his model, costs of subsidies can be covered using a lump sum tax. Our model has a fixed revenue requirement and no lump sum tax, so we study the tax design tradeoffs in prior second-best tax literature.

To summarize our five contributions, we first apply Corlett-Hague logic to production or disposal choices where ugly illegal emissions or disposal cannot be taxed. Second, we build a new analytical GE model to find closed-form solutions for changes in prices and quantities from a change in a tax rate. Third, we use those expressions in the welfare function to find closed-form solutions for the level of each SBO tax rate. Fourth, we interpret results to show how each outcome and each SBO tax rate depends on all three MED parameters, the substitution between the two goods in utility, all cross-price elasticities across waste options, and all input shares. Change in waste have ambiguous signs. Indeed, a change in tax on bad waste can lead to a rise or fall in either good waste or ugly waste. But our solutions include terms we can interpret. Only one of these terms is the “CH effect”, so we demonstrate other effects within our solutions.

Fifth, while our model lacks sufficient detail for specific policy analysis, we calibrate the model to illustrate concepts. We explore numerically the parameter space that yields the outcome where a tax on bad disposal is less than the tax on good disposal, and even where the tax on bad disposal becomes a subsidy (to avoid high damages of ugly disposal). Garbage and recycling collection often have a zero marginal price – specifically to avoid illegal dumping (NASEM, 2025). Thus, our model and derivations formalize what policymakers may already know.

For several reasons, our calibration is constructed to represent an approximation of any pollution problem (in the air, land, or water). First, tractability limits us to only one polluting sector with three disposal choices. Second, accounting data cannot show resources used in illegal disposal. Third, many parameters have never been estimated (own-price or especially cross-price elasticities). A specific calibration is not necessary to show that results are highly sensitive to elasticities that can vary over a wide range from negative to large positive values. Fourth, some marginal damages may be well measured (*e.g.*, Rode *et al.* 2021), but SBO tax rates depend on MEDs from all three alternatives, some of which may depend on population density and local circumstances. The MED from water pollutants are not well measured and must depend on how options are categorized. Measures of MED for recycling and for landfills are also not well measured, and the MED from dumping is not available (*e.g.*, Kinnaman 2016).

Moreover, any pollution problem can be represented by a particular choice of share and

elasticity parameters. We calculate SBO rates across substantial variation in parameters, so many different pollution problems can be represented by choosing a point on our sensitivity diagrams.

How do our results relate to prior results? We show how GE results differ significantly from simpler models. We also show how a bad pollutant might optimally be taxed more or less than a cleaner option. Importantly, however, we show the circumstances under which each such result pertains. We show which parameters are most important to estimate.

In prior literature with untaxed leisure, Bovenberg and Goulder (1996) calculate that the SBO tax on a polluting good can be higher or lower than its MED, depending on circumstances. They assume leisure is separable in utility, so all goods are equally substitutable for leisure. Using the logic of CH (1953), West and Williams (2007) estimate a cross-price elasticity between gasoline and leisure and “find that gasoline is a leisure complement, and thus that the optimal [SBO] gas tax exceeds the marginal external damage associated with gas use by roughly 35%” (p. 594). In our model where ugly waste is the untaxed choice, our calculations show how the SBO tax on bad waste disposal can be twice as high as its MED, or less than zero.

In our primary calibration, three disposal methods have similar elasticities but different damages. As expected, the SBO tax on bad disposal exceeds the SBO tax on good disposal. But if the cross-price elasticity of ugly disposal with respect to the price of bad disposal is doubled from 0.5 to 1.0 (or higher), and the MED from ugly disposal is three times the MED from bad disposal (or higher), then the SBO tax on good disposal exceeds the tax on bad disposal. Thus, optimal policy subsidizes bad behavior when worse ugly disposal (a) is highly damaging, (b) cannot be reduced directly, and (c) can be reduced by subsidizing an alternative.

The first section below describes our basic approach and welfare equations. Section 2 solves two special cases of our full GE model: Corlett-Hague with no externalities, and the Sallee model with lump-sum taxes. Section 3 describes our full GE model, its basic equations, and the linearization process. Section 4 finds effects of a small tax change on key outcomes, including the change in each type of disposal. Section 5 provides GE welfare analysis and finds closed-form solutions for the three FBO Pigovian tax rates and the two SBO tax or subsidy rates. Section 6 explains our calibration, and section 7 provides numerical results. Section 8 concludes.

1. Model Basics, Caveats, and Welfare Analysis

First, dynamics are ignored, in order to solve analytical GE models with production and disposal of multiple goods and disposal methods, each with its own cost, tax, price, and

substitutability. We compare two long-run equilibria, with and without a particular tax change. However, a dynamic model's shift in the short run can be less than in the long run. A subsidy to bad disposal may reduce ugly disposal but steer innovation towards bad technologies that persist. Still, our model can be used to compare outcomes assuming high or low cross-price elasticities.

Second, we study tax changes and not other policies. Because of perfect certainty, our intuition applies equally to a quantity regulation, where shadow prices on constraints match our tax rates. But regulations can be more complicated, and other policies are possible. Innovation policies matter, for example, such as a policy that can effectively increase the elasticities of substitution out of ugly disposal.

Third, a GE model could consider costly monitoring, so that second-best optimal enforcement is less than perfect. Policy tradeoffs might then impose a low tax rate on ugly disposal, to encourage greater compliance. But that tradeoff would make the analytics more difficult, and solutions more complex. It would not change the intuition behind the Corlett-Hague terms in our closed-form solutions for SBO taxes on good or bad disposal.

Fourth, we include three externalities but ignore other market failures well studied elsewhere (*e.g.*, other externalities, asymmetric information, and imperfect competition). In addition, international trade in waste can upend policy prescriptions in multiple ways.⁴

In our full GE model, firms produce X and Y in long-run equilibrium, where X creates waste for disposal types good G , bad B , or ugly U . Fixed total factors are used to produce goods and services (X, Y, G, B and U). In our closed economy with competition, n identical households, and constant returns to scale, all quantities can be scaled to amounts per household.

First, we solve two special cases in existing papers, represented here by removing X and Y , and assuming constant producer prices, no endogenous income, and no resource constraint. Each price paid is the sum of a cost per unit c_i and potential tax per unit τ_i (so $p_i = c_i + \tau_i$ for $i = G, B, U$). In these models, households maximize utility Ψ subject to a budget:

$$\mathcal{L} = \Psi[G, B, U; nG, nB, nU] + \lambda[I - p_G G - p_B B - p_U U] \quad (1.1)$$

Income I is the sum of exogenous income $\bar{I}$ and transfer of tax revenue back to households: $I = \bar{I} + T$. The number of households is assumed to be large enough that each household ignores its own contribution to the total public externalities (nG, nB, nU). For all models, we use a hat to

⁴ For example, Levinson (2023) points out that “Subsidies raise no concerns about leakage, because companies have no incentive to relocate overseas to avoid them.” (p.88).

define each proportional change, and we show how a shock to any one tax rate (*e.g.*, $\widehat{\tau}_B \equiv d\tau_B/\tau_B$) affects all market equilibrium quantities (*e.g.*, $\widehat{G} \equiv dG/G$).

Expressions for those equilibrium changes differ across models, as shown below, but the social planner's problem is the same for all models. **Appendix A** shows how the social planner maximizes welfare – differentiating the utility function but accounting for each household's effect on each total externality.⁵ The omniscient planner uses households' first-order conditions and their income constraint. Then divide by λ , the marginal utility of income, to get the dollar value of the change in utility ($d\Psi/\lambda$), and divide again by income I , to express this change as a fraction of income. The resulting change in welfare is:

$$\frac{d\Psi}{\lambda I} = (\tau_G - \mu_G) \frac{G}{I} \widehat{G} + (\tau_B - \mu_B) \frac{B}{I} \widehat{B} + (\tau_U - \mu_U) \frac{U}{I} \widehat{U} \quad (1.2)$$

where marginal external damage (MED) per unit is defined as: $\mu_G \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nG_X}$, $\mu_B \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nB_X}$, and $\mu_U \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nU_X}$. The minus sign in front of each MED defines damage as positive.⁶

Each model below shows how a small tax change leads to small changes in outcomes ($\widehat{G}$, $\widehat{B}$, and $\widehat{U}$). Equation (1.2) shows how those changes affect welfare. Each term has a net market failure (*e.g.*, $\tau_B - \mu_B$) times a change in quantity (*e.g.*, $\widehat{B}$). If that choice is “under-taxed” relative to its externality ($\tau_B < \mu_B$), then any tax change that raises B will reduce welfare in that term. If it is over-taxed, then any increase in quantity raises welfare in that term.

Assume second-order conditions hold, so welfare is at a maximum when $d\Psi = 0$. Then the first key result is clear. In any of our models with taxes possible on *all* pollutants, then the first-best optimal tax rates are $\tau_G = \mu_G$, $\tau_B = \mu_B$, and $\tau_U = \mu_U$ (as expected from Pigou, 1920). Substitution of those tax rates into (1.2) shows that welfare is maximized ($d\Psi = 0$).

As soon as the tax on ugly disposal becomes unavailable ($\tau_U = 0$), then each model has only two tax instruments to deal with three externalities. Then we solve for second-best optimal tax rates in levels, not changes. That is, we solve equation (1.2) not for effects of a shock such as $\widehat{\tau}_B$, but for the levels of pre-existing rates τ_B and τ_G – already in equation (1.2) – such that no change in any tax rate can raise welfare. In each model below, we solve for equilibrium changes ($\widehat{G}$, $\widehat{B}$, and $\widehat{U}$), plug those changes into equation (1.2), set $d\Psi = 0$, and solve for the levels of the

⁵ A later appendix shows that analogous derivations in the full GE model arrive at the same result as equation (1.2).

⁶ This procedure does not assume marginal damages are constant. The MED from additional disposal of each type might usually rise with the total amount of it.

two tax rates, which are the SBO rates (τ_G^* and τ_B^*).

2. Two Special Cases

In this section, we first show how the apparatus above can be used to demonstrate the original Corlett and Hague (1953) result, as presented in Auerbach and Hines (2002). Second, we replicate the result of the Sallee (2025).

For both models, we use equation (1.1) above, where the household's income-constrained maximization of utility results in three demands that can be written as functions of prices and incomes as on the left-hand sides of the following equations:

$$G = G(p_G, p_B, p_U, I) \Rightarrow \hat{G} = a_{GG}\hat{p}_G + a_{GB}\hat{p}_B + a_{GU}\hat{p}_U + \hat{I} \quad (2.1)$$

$$B = B(p_G, p_B, p_U, I) \Rightarrow \hat{B} = a_{BG}\hat{p}_G + a_{BB}\hat{p}_B + a_{BU}\hat{p}_U + \hat{I} \quad (2.2)$$

$$U = U(p_G, p_B, p_U, I) \Rightarrow \hat{U} = a_{UG}\hat{p}_G + a_{UB}\hat{p}_B + a_{UU}\hat{p}_U + \hat{I} \quad (2.3)$$

Appendix A also differentiates each demand on the left and re-arrange (*e.g.* divide both sides by quantity demanded). This procedure yields ($\Rightarrow$) a linear equation in small changes on each right-hand side of (2.3), where $a_{ij} \equiv \frac{\partial i}{\partial p_j} \frac{p_j}{i}$ is the elasticity of demand for disposal method i with respect to a change in price of j (for $i = G, B, U$, and holding all else constant). Then:

$$p_G = c_G + \tau_G \Rightarrow \hat{p}_G = t_G \hat{\tau}_G \quad (2.4)$$

$$p_B = c_B + \tau_B \Rightarrow \hat{p}_B = t_B \hat{\tau}_B \quad (2.5)$$

$$p_U = c_U + \tau_U \Rightarrow \hat{p}_U = t_U \hat{\tau}_U \quad (2.6a)$$

$$p_U = c_U \Rightarrow \hat{p}_U = 0 \quad (\text{if } \tau_U = 0) \quad (2.6b)$$

where $t_i \equiv \tau_i/p_i$ is the *ad valorem* tax rate for disposal method i (for $i = G, B, U$).

2.1 First Special Case

Corlett and Hague (1953) consider the case with a revenue requirement but no lump-sum tax (or equivalently, no tax on leisure in utility). We therefore include a revenue requirement, which we differentiate and re-arrange to get the linearized version on the right:

$$T = \tau_G G + \tau_B B \Rightarrow \hat{T} = \theta_{TG}(\hat{\tau}_G + \hat{G}) + \theta_{TB}(\hat{\tau}_B + \hat{B}) \quad (2.7)$$

where revenue shares are $\theta_{TG} \equiv \frac{\tau_G G}{T}$ and $\theta_{TB} \equiv \frac{\tau_B B}{T}$. We set $\hat{T} = 0$, so the total budget is fixed, as if a state or local waste disposal agency must pay for all disposal services using only disposal tax

revenue. We always assume each tax rate is on the normal side of its Laffer curve, so an increase in τ_B raises revenue that is used to reduce the other tax rate, τ_G .

Our models have no untaxed leisure, which is critical to virtually all of the second-best optimal tax literature mentioned in our footnote 2 above. Instead, we have untaxed ugly disposal and no lump-sum tax, which together generate an SBO tax problem with the same intuition. Nontaxation of U is a stand-in for nontaxation of leisure. If a more general model were also to include untaxed leisure, all results would be more complicated but would show the same cross-price effects within SBO taxes on good and bad disposal when positive revenue is required.

When one tax changes exogenously (*e.g.* $\widehat{\tau}_B > 0$), we have seven linear equations (2.1) to (2.7) to solve for seven unknowns: the $\widehat{\tau}_G$ that will balance the budget, three price changes ($\widehat{p}_G$, $\widehat{p}_B$, and $\widehat{p}_U$), and three quantity changes ($\widehat{G}$, $\widehat{B}$, and $\widehat{U}$). **Appendix B** inserts these quantity solutions into the welfare equation (1.2) with no externalities ($\mu_G = \mu_B = \mu_U = 0$) to find:

$$\frac{t_B}{t_G} = \frac{a_{GU} + a_{GB} + a_{BG}}{a_{BU} + a_{GB} + a_{BG}} \quad (2.8)$$

using *ad valorem* tax rates $t_G \equiv \tau_G/p_G$ and $t_B \equiv \tau_B/p_B$. This result matches the Corlett-Hague result in equation (3.21) of Auerbach and Hines (2002), if indices G and B are employed for their two taxable goods 1 and 2, and U replaces the index 0 for untaxed leisure. With a higher absolute size of the revenue requirement, both tax rates are higher, but in the same ratio.

This replication is useful not only to confirm that our more general model is built on the same foundation as the original SBO tax literature, but also to emphasize an interesting result: even with *no* externalities, the SBO tax is lower on whichever form of disposal is a better substitute for untaxed dumping. If the better substitute is bad disposal ($a_{BU} > a_{GU}$), then $t_B < t_G$. If so, then a low revenue requirement T can yield a positive τ_G^* and negative τ_B^* .

2.2 Second Special Case

The model in Sallee (2025) has two green goods and two untaxed-but-legal dirty goods (such as coal or gasoline). He studies effects of a “subsidy first” approach, such as in the climate provisions of the U.S. tax law of 2022. He finds a SBO rate for each taxable good that includes the MED of that good plus two more terms to correct for nontaxation of two dirty goods. Our model has only one untaxed good, so our replication finds a SBO rate for each taxable good that includes the MED of that good plus *one* more term to correct for nontaxation of one ugly good.

Our household problem in equation (1.1) can also represent Sallee’s problem with

externalities but no distorting revenue constraint. Any revenue loss is covered not by equation (2.7) but by an increase in lump-sum tax. **Appendix C** sets one exogenous tax change ($\widehat{\tau}_B > 0$) with no change in the other tax rate ($\widehat{\tau}_G = 0$). It then uses six equations in (2.1) to (2.6) to find six unknown changes in three prices ($\widehat{p}_G$, $\widehat{p}_B$, and $\widehat{p}_U$) and three quantities ($\widehat{G}$, $\widehat{B}$, and $\widehat{U}$). These quantity changes are inserted into the welfare equation (1.2) with the externalities that define our disposal problem ($0 < \mu_G < \mu_B < \mu_U$). The appendix derives the SBO rates:

$$\tau_G^* = \mu_G + \mu_U \frac{U(a_{UB}a_{BG} - a_{UG}a_{BB})}{G(a_{GB}a_{BG} - a_{GG}a_{BB})} \quad (2.9a)$$

$$\tau_B^* = \mu_B + \mu_U \frac{U(a_{UB}a_{GG} - a_{UG}a_{GB})}{B(a_{BB}a_{GG} - a_{BG}a_{GB})} \quad (2.9b)$$

Both Sallee's paper and ours express SBO rates in terms of MED values and cross-price effects, but he uses partial derivatives and we use elasticities. **Appendix C** shows that our equations (2.9) are equivalent to his equations (3) and (4) but without his extra term for the second untaxed dirty good. In our model with only one untaxed good, however, his assumptions lead to an interesting result. Appendix C shows that his rates for the two taxable goods (here labeled as G and B), reduce from equations (2.9) to the following:

$$\tau_G^* = \mu_G - \mu_U \quad \text{and} \quad \tau_B^* = \mu_B - \mu_U \quad (2.10)$$

Perhaps surprisingly, cross-price effects are now irrelevant. To explain, these tax rates are not second best. They are a first-best deposit-refund system, as demonstrated in Fullerton and Kinnaman (1995). A lump-sum tax is equivalent to a uniform tax on all goods in utility (all good, bad, *and* ugly). The lump-sum tax implicitly collects tax (deposit) at rate μ_U on all three goods; then each rate in (2.10) simply taxes each good at its own MED and returns the deposit (μ_U).

This analysis may help clarify matters, but his solution is still valid. It is not second best because of an untaxed good, but because of *two* untaxed goods. The lump-sum tax must apply to both at the same rate. If the two dirty goods in his model have the same MED (or nearly same MED), then his solution with two untaxed dirty goods is first best (or nearly first best).

Lump-sum taxes are not actual policy options, and they can lead to these simple first-best policies. Therefore, we limit the rest of our analysis to study of SBO policies with no tax on ugly waste and no lump-sum tax. Next, we present our full GE model, derive closed-form solutions, interpret them, calibrate the model, show sensitivity to parameters, and compare SBO tax rates from our GE results to the simpler results in the special cases above.

3. Our Analytical General Equilibrium Model

The only choices in the special cases above are good, bad, and ugly waste. Our GE model embeds these choices into a more general model that includes externalities, a revenue constraint, an economy-wide resource constraint, and two outputs: X includes goods that pollute or need disposal, and Y represents clean goods. Adding this choice allows another form of pollution abatement – not just substituting into a cleaner pollution or disposal method but also switching away from the polluting output altogether. Thus, n identical households maximize:

$$\mathcal{L} = \Psi[X, Y; nG, nB, nU] + \lambda[I - p_X^F X - p_Y Y], \quad (1.1')$$

where $X = W(G, B, U)$; $p_X^F X = p_X X + p_W X$; and $p_W X = p_G G + p_B B + p_U U$

The waste function $X = W(G, B, U)$ has two interpretations. In the first, households buy X at price p_X , knowing that each unit of X will also create a unit of waste W requiring additional disposal costs. Feasible combinations of the three disposal methods are represented by a three-dimensional iso-quant, with marginal rates of technical substitution between methods (*e.g.*, recycling, garbage, or dumping). Each method i costs p_i (for $i = G, B, U$) in equations (2.4-2.6), so the weighted average cost per unit of waste is p_W . The full unit price of X is $p_X^F = p_X + p_W$.

In the second interpretation, the same equations represent a nested Leontief function where production of each unit of X creates one unit of waste by-product, W , which is a flexible function of three polluting methods. Feasible combinations are represented by the same three-dimensional iso-quant (*e.g.*, methane capture, flaring, or venting/leaking). In this interpretation, the household pays p_X^F at the time of purchase, while the firm's cost is $p_X + p_W$. Either way, the firm or household chooses the amount of each kind of disposal to minimize total costs.

Firms produce output using a single primary input K , a composite of capital, labor, and other primary inputs. It is perfectly mobile, has a single price p_K , and is in fixed total supply $\bar{K}$. Production is given by $X = X(K_X)$ and $Y = Y(K_Y)$. Competition and constant returns to scale (CRTS) allow units defined per household and imply that production can be $X = K_X/c_X$ and $Y = K_Y/c_Y$, where c_X and c_Y are each a fixed use of primary factor per unit of output.

Each basic equation is differentiated and re-arranged to yield ($\Rightarrow$) a linear equation in hats (*e.g.*, $\hat{X} \equiv dX/X$). Derivations are in **Appendix D**. The key equations are numbered, for the later use of N independent linear equations to solve for N unknown changes in prices and quantities:

$$X = K_X/c_X \quad \Rightarrow \quad \hat{X} = \hat{K}_X \quad (3.1)$$

$$p_X X = p_K K_X \quad \Rightarrow \quad \widehat{p}_X + \widehat{X} = \widehat{p}_K + \widehat{K}_X \quad (3.2)$$

$$Y = K_Y / c_Y \quad \Rightarrow \quad \widehat{Y} = \widehat{K}_Y \quad (3.3)$$

$$p_Y Y = p_K K_Y \quad \Rightarrow \quad \widehat{p}_Y + \widehat{Y} = \widehat{p}_K + \widehat{K}_Y \quad (3.4)$$

Insert equation (3.1) into equation (3.2) to get $p_X = p_K c_X$, since only K is used to produce X .

The implied linear equation is $\widehat{p}_X = \widehat{p}_K$ (which is not a numbered independent linear equation because it is derived from the first two). Analogously, these equations imply $\widehat{p}_Y = \widehat{p}_K$.

Each kind of waste has its own MED defined as μ_i ($i = G, B, U$). We assume $0 < \mu_G < \mu_B < \mu_U$, to fit our definitions.⁷ Households face private costs of disposal, including the cost of their own garbage and recycling bins, space in their garage to store them, and the time they take for daily disposal and weekly transport to the curb. We define private costs by fixed parameters c_G per unit of good disposal, c_B per unit of bad disposal, and c_U per unit of ugly disposal. Then:

$$K_G = c_G G \quad \Rightarrow \quad \widehat{K}_G = \widehat{G} \quad (3.5)$$

$$K_B = c_B B \quad \Rightarrow \quad \widehat{K}_B = \widehat{B} \quad (3.6)$$

$$K_U = c_U U \quad \Rightarrow \quad \widehat{K}_U = \widehat{U} \quad (3.7)$$

$$\bar{K} = K_X + K_Y + K_G + K_B + K_U \quad \Rightarrow \quad (3.8)$$

$$0 = \widehat{K}_X \theta_{KX} + \widehat{K}_Y \theta_{KY} + \widehat{K}_G \theta_{KG} + \widehat{K}_B \theta_{KB} + \widehat{K}_U \theta_{KU}$$

Equation (3.5) implies that the per-unit cost of producing G is $p_K c_G$. Also, equations (3.6) and (3.7) imply that $p_K c_B$ (and $p_K c_U$) is the unit cost of B (and U).⁸ Equation (3.8) is the economy's resource constraint (per household). Other equations below require similar ratio or share parameters, so linearization uses θ_{ji} to represent the fraction of the first subscript (j) attributable to the second subscript (i). Here, for example, $\theta_{Ki} \equiv K_i / \bar{K}$ is the fraction of $\bar{K}$ attributable to K_i (for $i = X, Y, G, B, U$), and $\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB} + \theta_{KU} = 1$.

The relevant policy variables are positive or negative tax rates: τ_G per unit of good waste, τ_B per unit of bad, and τ_U per unit of ugly (for the simple case used to show our model matches Pigou's FBO solution). The household's full cost of good disposal ($p_G G$) equals the sum of

⁷ Even good disposal such as recycling is not harmless and has associated environmental damages from collection, transportation, and processing. Analogously, even the "Good" portrayed by Clint Eastwood in the 1966 film causes damage despite his name.

⁸ An alternative interpretation of the same equations is that the household pays haulers to collect G and B , firms with CRTS production functions $G = K_G / c_G$ and $B = K_B / c_B$. Competitive prices would match their per-unit costs of production, $p_K c_G$ and $p_K c_B$. Thus, in this interpretation, consumer costs are the same as in the text.

private cost ($p_K c_G G$) and tax cost ($\tau_G G$). We then define $\theta_{Gp} \equiv p_K c_G / (p_K c_G + \tau_G)$ for the private share of the cost and $\theta_{G\tau} \equiv \tau_G / (p_K c_G + \tau_G)$ for the tax share (where $\theta_{Gp} + \theta_{G\tau} = 1$).

With similar definitions for the other inputs, we have:

$$p_G G = p_K c_G G + \tau_G G \Rightarrow \widehat{p}_G + \widehat{G} = \theta_{Gp}(\widehat{p}_K + \widehat{G}) + \theta_{G\tau}(\widehat{\tau}_G + \widehat{G}) \quad (3.9)$$

$$p_B B = p_K c_B B + \tau_B B \Rightarrow \widehat{p}_B + \widehat{B} = \theta_{Bp}(\widehat{p}_K + \widehat{B}) + \theta_{B\tau}(\widehat{\tau}_B + \widehat{B}) \quad (3.10)$$

$$p_U U = p_K c_U U + \tau_U U \Rightarrow \widehat{p}_U + \widehat{U} = \theta_{Up}(\widehat{p}_K + \widehat{U}) + \theta_{U\tau}(\widehat{\tau}_U + \widehat{U}) \quad (3.11a)$$

$$\text{or } p_U U = p_K c_U U \Rightarrow \widehat{p}_U + \widehat{U} = \widehat{p}_K + \widehat{U} \quad \text{with fixed } \tau_U = 0 \quad (3.11b)$$

$$p_W X = p_G G + p_B B + p_U U \Rightarrow \widehat{X} + \widehat{p}_W = \quad (3.12)$$

$$\theta_{XG}(\widehat{p}_G + \widehat{G}) + \theta_{XB}(\widehat{p}_B + \widehat{B}) + \theta_{XU}(\widehat{p}_U + \widehat{U})$$

$$p_X^F X = p_X X + p_W X \Rightarrow \widehat{p}_X^F + \widehat{X} = \theta_{pX}(\widehat{p}_X + \widehat{X}) + \theta_{pW}(\widehat{p}_W + \widehat{X}) \quad (3.13)$$

Bad disposal in equation (3.10) is analogous to (3.9). For ugly disposal, (3.11a) assumes $\tau_U \neq 0$, so it is similar to (3.9) and (3.10). Then (3.11b) fixes $\tau_U = 0$ (for SBO solutions).⁹ In this case, $\widehat{p}_U = \widehat{p}_K$. The quantity of X matches the quantity of waste, so equation (3.12) defines the overall cost of waste disposal p_W per unit of X . To arrive at (3.12), add the cost of good, $p_G G$, bad, $p_B B$, and ugly, $p_U U$ (e.g., transporting waste to the woods for dumping). In the linearized equation (3.12), we define $\theta_{XG} \equiv \frac{p_G G}{p_W X}$, $\theta_{XB} \equiv \frac{p_B B}{p_W X}$, and $\theta_{XU} \equiv \frac{p_U U}{p_W X}$ (where the three sum to one). Then equation (3.13) defines a household's full spending on X as $p_X^F X \equiv p_X X + p_W X$. Divide both sides by X to get the basic equation for the full price p_X^F on the left side of (3.13). The linearized equation (3.13) defines $\theta_{pX} \equiv \frac{p_X}{p_X^F}$ and $\theta_{pW} \equiv \frac{p_W}{p_X^F}$, where $\theta_{pX} + \theta_{pW} = 1$.

Given how this full price p_X^F varies with disposal costs, the consumer can substitute from X into Y if an increase in tax τ_G raises p_G (which raises p_X^F by an amount that depends on the good disposal share θ_{XG}). Similarly, the household can consume more of good X and less of Y in response to a larger subsidy τ_B (depending on the bad disposal share θ_{XB}).

Differentiation of the waste equation yields the next linear equation, where expenditure shares are $\theta_{XG} \equiv \frac{p_G G}{p_W X}$, $\theta_{XB} \equiv \frac{p_B B}{p_W X}$, and $\theta_{XU} \equiv \frac{p_U U}{p_W X}$ (with $\theta_{XG} + \theta_{XB} + \theta_{XU} = 1$):

⁹ We could weaken the policy constraint to say that τ_U is not fixed at zero but at a sub-optimally positive level. That generalization would have little effect on our results below about how SBO tax rates are determined by elasticities, MED parameters, and share parameters. The main effect would be to complicate aesthetics of solutions by adding more terms, because total revenue $T = \tau_G G + \tau_B B + \tau_U U$ would be affected by changes in U . Below, we assume total tax revenue is fixed, so a small change in τ_B is offset by an opposite change in τ_G .

$$X = W(G, B, U) \quad \Rightarrow \quad \hat{X} = \theta_{XG}\hat{G} + \theta_{XB}\hat{B} + \theta_{XU}\hat{U} \quad (3.14)$$

$$\hat{G} - \hat{U} = \theta_{XG}(e_{GG} - e_{UG})\hat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\hat{p}_B + \theta_{XU}(e_{GU} - e_{UU})\hat{p}_U \quad (3.15)$$

$$\hat{B} - \hat{U} = \theta_{XG}(e_{BG} - e_{UG})\hat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\hat{p}_B + \theta_{XU}(e_{BU} - e_{UU})\hat{p}_U. \quad (3.16)$$

For choices among inputs, **Appendix D** derives equations (3.15) and (3.16) following Allen (1938), Mieszkowski (1972), and Fullerton and Heutel (2007). Equations (2.1) to (2.3) define $a_{ij} \equiv \frac{\partial i}{\partial p_j} \frac{p_j}{i}$ as the elasticity of demand for input i with respect to the price of j (holding all else constant). Allen (1938) shows that $a_{ij} = e_{ij}\theta_{Xj}$, where e_{ij} is the Allen partial elasticity of substitution, and θ_{Xj} is the share for input j . Allen also shows that $a_{iG} + a_{iB} + a_{iU} = 0$ and that $e_{ii} < 0$. Uzawa (1962) shows symmetry ($e_{ij} = e_{ji}$). The elasticity e_{ij} is positive for substitutes and negative for complements. At most one of three cross-price e_{ij} can be negative.

Pre-existing disposal tax rates yield tax revenue T per household in equation (3.17), where differentiation uses shares $\theta_{TG} \equiv \tau_G G/T$ and $\theta_{TB} \equiv \tau_B B/T$ (which sum to one). In the case where ugly disposal cannot be taxed, we have:

$$T = \tau_G G + \tau_B B \quad \Rightarrow \quad \hat{T} = \theta_{TG}(\hat{\tau}_G + \hat{G}) + \theta_{TB}(\hat{\tau}_B + \hat{B}) \quad (3.17)$$

$$I = p_K \bar{K} + T \quad \Rightarrow \quad \hat{I} = \frac{p_K \bar{K}}{I} \hat{p}_K + \frac{T}{I} \hat{T} \quad (3.18)$$

We set $\hat{T} = 0$, so (3.17) implies a fixed revenue that is added to household budgets. Thus we can address our key question about the two SBO tax rates relative to each other and avoid conflating economic effects of a tax revenue increase with other economic effects from use of extra revenue. We avoid a change in public spending that would have its own economic effects; we avoid a change in another distorting tax that would greatly complicate our second-best analysis; and we avoid a change in any pre-existing lump-sum tax or transfer.

Finally, household utility is continuous, quasi-concave, and twice differentiable. It is increasing in consumption goods X and Y , and decreasing in each of the three total disposal amounts, *i.e.*, public bads that are separable in utility. Then the linear equation in (3.19) defines σ_ψ as the elasticity of substitution between X and Y in utility.

$$\Psi = \Psi(X, Y; nG, nB, nU) \quad \Rightarrow \quad \hat{X} - \hat{Y} = \sigma_\psi (\hat{p}_Y - \hat{p}_X^F) \quad (3.19)$$

The household maximizes utility subject to its budget. If the household chooses disposal methods, then the disposal function $X = W(G, B, U)$ can be substituted into the utility function.

Then each household effectively chooses only G, B, U , and Y .

4. General Equilibrium Outcomes with Three Polluting Inputs

We have ten unknown quantities ($\hat{X}, \hat{K}_X, \hat{Y}, \hat{K}_Y, \hat{G}, \hat{K}_G, \hat{B}, \hat{K}_B, \hat{U}$, and $\hat{K}_U$), eight prices ($\hat{p}_K, \hat{p}_Y, \hat{p}_X, \hat{p}_B, \hat{p}_G, \hat{p}_U, \hat{p}_W$, and $\hat{p}_X^F$), two tax rates ($\hat{\tau}_B$ and $\hat{\tau}_G$), and two other values ($\hat{T}$ and $\hat{I}$). Of those 22 variables, we set one numeraire ($\hat{p}_K = 0$), and one exogenous policy shock (e.g., $\hat{\tau}_G > 0$ or $\hat{\tau}_B > 0$). We assume revenue is fixed ($\hat{T} = 0$), so a policy shock $\hat{\tau}_B > 0$ raises revenue to reduce the other tax rate ($\hat{\tau}_G < 0$). Thus, the number of unknowns falls from 22 to 19, and we can solve for all outcomes using the 19 numbered equations (3.1) to (3.19).

4.1 Solutions for Changes in the Good, the Bad and the Ugly

To solve for SBO tax rates in a later section below, we first start from any (non-optimal) initial set of tax rates (τ_G and τ_B), assume an exogenous increase in tax on bad disposal ($\hat{\tau}_B$), and solve for changes in disposal ($\hat{G}, \hat{B}, \hat{U}$). **Appendix E** starts these derivations with an intermediate equation (not yet closed-form solution, because it includes an endogenous change $\hat{\tau}_G$):

$$\hat{U} = \varepsilon_{UB}\hat{\tau}_B + \varepsilon_{UG}\hat{\tau}_G \quad (4.1)$$

where $\varepsilon_{UB} \equiv [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi \theta_{pW} \theta_{KY}] \theta_{XB} \theta_{B\tau}$ and $\varepsilon_{UG} \equiv [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_\psi \theta_{pW} \theta_{KY}] \theta_{XG} \theta_{G\tau}$.

The Allen elasticity e_{ij} is the PE effect on i for a change in price j . Our results below use E_{ij} to denote the full GE effect on i for a change in price j , but the ε_{ij} are neither PE nor GE. The first term of (4.1) uses ε_{UB} for the change in U from an exogenous change in τ_B without accounting for the endogenous $\hat{\tau}_G$. Analogously, the second term uses ε_{UG} as the change in U from a change only in τ_G without accounting for any change in τ_B . Both are composites that depend upon the e_{ij} and cost shares (the θ 's).

The logic of Corlett and Hague (CH, 1953) would compare ε_{UB} and ε_{UG} to see whether ugly disposal responds more to the tax on bad disposal or on good disposal. As equation (4.1) makes clear, the effect on ugly waste does not just compare ε_{UB} vs. ε_{UG} . It also depends on expenditure shares, the revenue constraint, and the size of each tax rate change. In each of the first three terms of ε_{UB} and ε_{UG} , the change in each tax also has indirect effects on the other taxable disposal. The fourth term in both ε_{UB} and ε_{UG} is $-\sigma_\psi \theta_{pW} \theta_{KY}$, because an increase in either tax rate raises overall costs of disposal, p_W , and thus raises the full price to buy more X .

The elasticity σ_ψ indicates how households substitute out of X into Y . That substitution reduces X , so it reduces all forms of disposal (including ugly, via this fourth term).

We expect effects in the first terms containing e_{UB} and e_{UG} to outweigh other indirect effects, but perverse outcomes are possible. Later, our calibration indicates the relative size of each effect. For now, solutions for the other changes in disposal are:

$$\widehat{B} = \varepsilon_{BB}\widehat{\tau}_B + \varepsilon_{BG}\widehat{\tau}_G \quad (4.2)$$

where $\varepsilon_{BB} \equiv [e_{UB}(-\theta_{KU}) + e_{BB}(1 - \theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}$ and

$\varepsilon_{BG} \equiv [e_{UG}(-\theta_{KU}) - e_{GG}(\theta_{KG}) + e_{BG}(1 - \theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}$.

$$\widehat{G} = \varepsilon_{GB}\widehat{\tau}_B + \varepsilon_{GG}\widehat{\tau}_G \quad (4.3)$$

where $\varepsilon_{GB} \equiv [e_{UB}(-\theta_{KU}) - e_{BB}(\theta_{KB}) + e_{GB}(1 - \theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}$ and

$\varepsilon_{GG} \equiv [e_{UG}(-\theta_{KU}) + e_{GG}(1 - \theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}$.

In Equation (4.2), ε_{BB} is the intermediate elasticity for the percent change in B from a one percent change only in τ_B , and ε_{BG} is the percent change in B from a one percent change only in τ_G . In equation (4.3), ε_{GB} and ε_{GG} are analogous elasticities for the changes in good disposal. All intermediate elasticities contain direct and indirect PE elasticities and expenditure shares analogous to those described earlier for how tax rates affect U .

4.2 Solving for $\widehat{\tau}_G$ in Terms of Parameters and Exogenous $\widehat{\tau}_B$

With a shock $\widehat{\tau}_B > 0$ and the government revenue constraint in equation (3.17) as well as the changes to each tax base in (4.2) and (4.3), **Appendix F** shows how we obtain:

$$E_{TG}\widehat{\tau}_G = -E_{TB}\widehat{\tau}_B \quad (4.4)$$

where $E_{TG} \equiv \theta_{TG} + \theta_{TG}\varepsilon_{GG} + \theta_{TB}\varepsilon_{BG}$ and $E_{TB} \equiv \theta_{TB} + \theta_{TB}\varepsilon_{BB} + \theta_{TG}\varepsilon_{GB}$.

That Appendix then uses (4.4) to solve for $\widehat{\tau}_G$, and it shows that E_{TG} is the GE elasticity of total tax revenue T for a change in the tax rate on good disposal τ_G . Analogously, E_{TB} is the GE elasticity for the change in T from a change in τ_B . Equation (4.4) requires that the revenue loss from the tax cut ($E_{TG}\widehat{\tau}_G$) matches the revenue increase ($E_{TB}\widehat{\tau}_B$).

The first term of E_{TB} is θ_{TB} , the initial share of tax revenue from τ_B . When multiplied by $\widehat{\tau}_B > 0$, it is the revenue from raising the tax rate on the existing tax base (B). The second term within E_{TB} is $\theta_{TB}\varepsilon_{BB}$, which captures the fact that $\widehat{\tau}_B > 0$ reduces the tax base B (reducing revenue). Finally, the third term includes ε_{GB} , the percent change in G from a one percent change

in τ_B .¹⁰ It is positive when ε_{GB} positive, because it is multiplied by $\widehat{\tau}_B > 0$ and by the share of revenue from the other tax base ($\theta_{TG} > 0$). This third term is the indirect increase in total revenue from substitution to more good disposal (which is also taxed).

Parameters within E_{TG} are analogous. On the normal sides of both Laffer curves, both GE revenue elasticities E_{TG} and E_{TB} are positive. If $E_{TB} > E_{TG}$, then total tax revenue increases more in response to a one percent change in τ_B than to a one percent change in τ_G .

4.3 Closed-form Solutions for Changes in Disposal Amounts

Equations (4.1)–(4.3) show changes in the three kinds of disposal, but each depends on the endogenous change in the other tax rate. Here, we solve for these outcomes in terms only of exogenous parameters and the exogenous shock $\widehat{\tau}_B$. Full derivations are in **Appendix G**.

For closed-form solutions, we substitute (4.4) into equations (4.1), (4.2), and (4.3) to get:

$$\widehat{G} = E_{GB}\widehat{\tau}_B \quad \text{where} \quad E_{GB} \equiv \varepsilon_{GB} - \varepsilon_{GG} \frac{E_{TB}}{E_{TG}} \quad (4.5)$$

$$\widehat{B} = E_{BB}\widehat{\tau}_B \quad \text{where} \quad E_{BB} \equiv \varepsilon_{BB} - \varepsilon_{BG} \frac{E_{TB}}{E_{TG}} \quad (4.6)$$

$$\widehat{U} = E_{UB}\widehat{\tau}_B \quad \text{where} \quad E_{UB} \equiv \varepsilon_{UB} - \varepsilon_{UG} \frac{E_{TB}}{E_{TG}} \quad (4.7)$$

Here, we define E_{iB} as the full GE elasticity for the percentage change in disposal i from a one percent change in the tax on the bad disposal τ_B .

In our discussion above, the logic of CH compares the two key PE cross-price elasticities e_{UB} and e_{UG} . In the closed-form solution of equation (4.7), the single coefficient (E_{UB}) on the exogenous shock ($\widehat{\tau}_B$) makes exactly that comparison of elasticities. The definition of E_{UB} starts with ε_{UB} , defined to include e_{UB} , and it subtracts ε_{UG} , defined to include e_{UG} . Next, we use the three disposal *changes* in (4.5)–(4.7) to solve for the *level* of each FBO and SBO tax rate.

5. Welfare Analysis in Our GE Model

This section follows the same logic as in the welfare derivations in Section 1 above, but for our GE model. We then solve for second-best optimal tax rates

First, substitute the waste function $X = W(G, B, U)$ into the utility function, so that the household can maximize a Lagrangean by choice of Y and disposal methods G, B, U :

¹⁰ Within ε_{GB} of equation (4.3), an exogenous $\widehat{\tau}_B > 0$ affects G most directly through a positive term $e_{GB}(1 - \theta_{KG})$, thereby expanding tax revenue collected from G . Two other terms within ε_{GB} are $e_{UB}(-\theta_{KU})$ and $-\sigma_\Psi \theta_{PW} \theta_{KY}$. These terms are negative, so the effect on G is not unambiguously positive.

$$\mathcal{L} = \Psi[W(G, B, U), Y; nG, nB, nU] + \lambda[I - p_X^E X - p_Y Y]$$

As shown in **Appendix H**, the first order conditions from this household maximization are used by the social planner along with the economy-wide resource constraint in equation (3.8), all inserted into the differentiated utility function. Then divide by the marginal utility of income to arrive at the dollar value of the change in utility ($d\Psi/\lambda$) and divide both sides by income I . The Appendix shows that the resulting change in welfare is exactly the same as in equation (1.2) for simpler models. As before, the FBO rates are $\tau_G = \mu_G$, $\tau_B = \mu_B$, and $\tau_U = \mu_U$. Substitution of those tax rates into (1.2) shows that welfare is maximized ($d\Psi = 0$).

5.1 SBO Solutions with No Tax on Ugly Disposal

Steps are listed here to find SBO tax rates when $\tau_U = 0$, but **Appendices I and J** provide full derivations. First, for tax shock $\widehat{\tau}_B$, insert equations (4.5)-(4.7) into (1.2) and set $d\Psi = 0$ to get one equation in two initial tax rates (τ_G and τ_B). Replace the τ_G with $(T - \tau_B B)/G$ from equation (3.17). Solve the resulting equation for τ_B to show what must be the initial τ_B in equation (1.2) such that any change in disposal ($\widehat{G}$, $\widehat{B}$, or $\widehat{U}$) does not change welfare ($d\Psi = 0$).

Second, for tax shock is $\widehat{\tau}_G$, we repeat all steps described above to solve all equations (3.1)-(3.19), to get new versions of (4.5)-(4.7) to insert into (1.2) to get a new equation in both initial tax rates τ_G and τ_B . Solve (3.17) for τ_B , and use it to get a new equation for the initial τ_G where $d\Psi = 0$. The two resulting SBO tax rates are:

$$\tau_B^* = \frac{\mu_B + \underbrace{\mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)}_{\text{Corlett-Hague Term}} + \underbrace{\left(\mu_G - \frac{T}{G} \right) \frac{G}{B} \left(\frac{E_{GB}}{E_{BB}} \right)}_{\text{Revenue Term}}}{1 - E_{GB}/E_{BB}} \quad (5.1)$$

$$\tau_G^* = \frac{\mu_G + \underbrace{\mu_U \frac{U}{G} \left(\frac{E_{UG}}{E_{GG}} \right)}_{\text{Corlett-Hague Term}} + \underbrace{\left(\mu_B - \frac{T}{B} \right) \frac{B}{G} \left(\frac{E_{BG}}{E_{GG}} \right)}_{\text{Revenue Term}}}{1 - E_{BG}/E_{GG}} \quad (5.2)$$

where $E_{GG} \equiv \varepsilon_{GG} - \varepsilon_{GB} \frac{E_{TG}}{E_{TB}}$, $E_{BG} \equiv \varepsilon_{BG} - \varepsilon_{BB} \frac{E_{TG}}{E_{TB}}$, and $E_{UG} \equiv \varepsilon_{UG} - \varepsilon_{UB} \frac{E_{TG}}{E_{TB}}$. Each E_{ij} is a GE change in i from a one percent change in τ_j . Overall, SBO tax rates depend on MED parameters

(μ_G , μ_B , and μ_U), initial polluting input levels (*e.g.*, G/B), and all GE elasticities.¹¹

5.2 Detailed Discussion of SBO Tax Rates

Both equations have a numerator with three key terms, and a denominator equal to one minus a ratio with a GE cross-price elasticity E_{ij} that could have either sign. Again, we limit discussion to the case where all types of disposal are substitutes ($E_{ij} > 0$). This E_{ij} is divided by an own-price elasticity that must be negative, so each denominator is positive and larger than one. Thus, it scales down every term in the numerator but does not change any sign.

In the numerator of equation (5.1) for τ_B^* , the first term is the “own damage” term (μ_B reduced by a denominator larger than one). The other terms tell the social planner more about how to deviate from the FBO tax rate (μ_B).

The second term in (5.1) is the “Corlett-Hague term”, adjusting τ_B^* to account for damage from its effect on ugly disposal. It depends on the ugly MED (μ_U), the initial relative amounts of disposal (U/B), and GE effect of τ_B^* on U (E_{UB} , presumed positive). This GE effect is divided by $E_{BB} < 0$, which means that the whole term is negative. Thus, this second term is a reduction in the planner’s τ_B^* , intended to encourage switching toward pollution that is merely bad rather than ugly. Similarly, the second term in (5.2) says τ_G^* is reduced to account for damage from its effect on U , which depends on E_{UG} . However, the revenue constraint means that these CH terms are determined relative to each other. In other words, the CH term only subtracts more from τ_B^* than from τ_G^* if the impact of τ_B^* on U is bigger than the impact of τ_G^* on U .

The third term in (5.1) is a “revenue term” that accounts for the GE effect of τ_B^* on good disposal (via E_{GB}), which affects revenue because good disposal is taxed.¹² This term derives from the $(\tau_G - \mu_G) \frac{G}{I} \hat{G}$ that appears in the welfare change equation (1.2), where $\hat{G}$ is multiplied not by full damages μ_G , but only by uncorrected damages ($\mu_G - \tau_G$). In that term, if τ_G happens to equal μ_G , then any effect of $\hat{\tau}_B$ on good disposal would have no impact on social welfare (so this term does not modify τ_B^*). If $\tau_G > \mu_G$, then the externality is overcorrected, and the planner adjusts τ_B^* upward to encourage substitution into the over-taxed good disposal.

¹¹ Each GE elasticity is a calculated combination of other intermediate elasticities (ε_{ij}), each of which is a calculated combination of fixed PE elasticity parameters (all of the e_{ij}).

¹² Equations (4.1)–(4.3) show how each GE elasticity depends on Allen elasticities e_{ij} and share parameters θ . For example, the GE elasticity of good disposal with respect to the price of bad disposal depends on all e_{ij} , because any one tax also leads to GE feedback effects on the other two methods of disposal.

In the revenue term of τ_B^* in equation (5.1), however, the revenue constraint (3.17) means that $(\mu_G - \tau_G)$ is replaced by $(\mu_G - \frac{T}{G})$. This change replaces the tax per unit τ_G with a calculated tax per unit that could collect *all* needed revenue T (as if τ_B were zero). Similarly, τ_G^* in (5.2) includes the analogous revenue term, so the two revenue terms together tell the planner how to set both τ_B^* and τ_G^* simultaneously to achieve the best tradeoff between good and bad disposal. Also, the revenue requirement T is arbitrary. Together, (5.1) and (5.2) make clear that a larger T would require higher values of both SBO tax rates.

5.3 Broad Discussion

These SBO tax rate equations use intuition from Corlett and Hague (1953), but not their simpler model. In our GE model, the CH logic is seen by comparing the CH-term of τ_B^* in (5.1) to the CH-term of τ_G^* in (5.2), but this comparison is complicated by five factors.

First, cross-price effects in our model are not based directly on PE cross-price elasticities e_{UB} and e_{UG} but on all GE elasticities E_{ij} , each of which depends on *all* own- and cross-price e_{ij} and share parameters θ_{ij} . In other words, SBO tax rates depend on GE effects of a disposal tax that account for all changes in allocations (*not* holding all else constant).

Second, relative to CH, we add external damages. SBO tax rates τ_B^* and τ_G^* depend importantly on the MED from ugly disposal, μ_U (especially as compared to the other MED).

Third, each SBO tax rate also depends on the MED of the other *taxable* method of disposal, and on the GE cross-price elasticity between the two taxable methods of disposal (comparing the two “revenue terms” in our SBO solutions).

Fourth, those factors just listed do not determine the absolute size of the two tax rates. We find the extent to which each tax rate is adjusted away from the MED of the disposal method being taxed, but the overall size of both tax rates depend on the revenue requirement.

Fifth, further analytical work could take the derivative of each SBO tax rate with respect to any key parameter (*e.g.*, μ_U). That procedure might appear to show that τ_B^* and τ_G^* fall with a higher MED of ugly, since μ_U enters each tax rate negatively and only once in the numerator. But our revenue constraint means that a higher μ_U cannot reduce both tax rates. This constraint is built into GE elasticities, so the effect of μ_U on a tax rate is not obvious. All parameters matter for GE outcomes. We cannot take a derivative of (5.1) or (5.2) with respect to a parameter such as μ_U , because it appears many times in the numerator and denominator of each SBO tax. Thus,

we turn to calibration and sensitivity analysis for the effect of each parameter on each tax rate.

6. Calibration

Our choices for “primary” parameter values are shown in **Table 2**. For this hypothetical closed economy, total income is 1.0. For example, if this income refers to 1.0 trillion dollars, then the first column says that \$350 billion of primary factors are used to produce X , and \$350 billion are used to produce Y (before any tax shock). Also, \$100 billion is used to undertake each kind of disposal (where each is one-third of total waste costs in the second column). The initial tax share is assumed to be 10% of disposal cost for each taxable disposal method (so the private cost is the other 90%). The elasticity of substitution between X and Y is assumed to be one.

Table 2: Initial Equilibrium Parameter Values

Assumed Parameter Values		Calculated in Appendix K Parameter Values		Assumed Parameter Values	
θ_{KX}, θ_{KY}	0.35	$\theta_{XG}, \theta_{XB}, \theta_{XU}$	0.333	μ_G	0.05
$\theta_{KG}, \theta_{KB}, \theta_{KU}$	0.1	θ_{pX}	0.538	μ_B	0.15
$\theta_{G\tau}, \theta_{B\tau}$	0.1	θ_{pW}	0.462	μ_U	0.5
θ_{Gp}, θ_{Bp}	0.9	τ_G, τ_B	0.111	e_{UG}, e_{UB}	0.5
σ_ψ	1	θ_{TG}, θ_{TB}	0.5	e_{GB}, e_{BG}	0.5
		T	0.0222	e_{GG}, e_{BB}, e_{UU}	-1

As shown in **Appendix K**, the assumptions in the first column imply parameter values in the second column. The two initial tax shares are 0.1 and quantities match, so the calculated tax rates are $\tau_G = \tau_B = 0.111$. Each raises half of total tax revenue, $T = 0.0222$.

In the third column, values for the MED are chosen such that $\mu_G < \mu_B < \mu_U$. Finally, all assumed Allen cross-price elasticities are 0.5, and each own-price elasticity is -1 .

In sensitivity analysis, each MED can be varied independently. That is, we can alter the external cost of ugly disposal μ_U without a change in any other parameter value. In addition, we want to vary the key cross-price elasticity e_{UB} , but some other parameter must change for the symmetrical Allen elasticity matrix to satisfy the adding up constraint for each row of the matrix, $e_{iG}\theta_{XG} + e_{iB}\theta_{XB} + e_{iU}\theta_{XU} = 0$ (for $i = G, B, U$). We want to show results for different values of e_{UB} , but without changes in other cross-price elasticities. Therefore, as our program varies e_{UB} , it calculates the change in own-price elasticities necessary to satisfy that constraint.

Next, **Table 3** maps the Allen elasticities e_{ij} into the a_{ij} elasticities used in the second

special case, and then into the intermediate and full GE elasticities E_{ij} , using definitions in equations (4.1)-(4.7) and the parameter values in Table 2. It shows that the GE elasticities can be quite different from the PE Allen elasticities, which points toward the importance of using a GE model. Our GE elasticities E_{ij} differ from PE elasticities e_{ij} because they incorporate all assumptions of the model, including: the fixed revenue requirement, the fixed economy-wide resource constraint, the many relative-size parameters θ_{ij} , and simultaneous changes to other prices (and in other models, endogenous changes to producer prices, incomes, and profits).

Table 3: Mapping Primary Allen PE Elasticities to Full GE Elasticities

Allen PE Elasticity	Assumed Value	$a_{ij} = e_{ij}\theta_{xj}$	Intermediate Elasticity	Calculated Value	Full GE Elasticity	Calculated Value
e_{UG}	0.5	0.167	ε_{UG}	0.011	E_{UG}	0
e_{UB}	0.5	0.167	ε_{UB}	0.011	E_{UB}	0
e_{GB}	0.5	0.167	ε_{GB}	0.011	E_{GB}	0.050
e_{BG}	0.5	0.167	ε_{BG}	0.011	E_{BG}	0.050
e_{GG}	-1	-0.333	ε_{GG}	-0.039	E_{GG}	-0.050
e_{BB}	-1	-0.333	ε_{BB}	-0.039	E_{BB}	-0.050
e_{UU}	-1	-0.333	ε_{UU}	-0.039	E_{TG}	0.486
					E_{TB}	0.486

We assume values for the e_{ij} and then calculate the corresponding a_{ij} for the second special case below.

Most important here is the revenue constraint, which means that an increase in one tax requires a cut in the other tax. The first rows of Table 3 show that all e_{ij} are 0.5 (when $i \neq j$), but GE elasticities E_{UG} and E_{UB} are zero (and other GE elasticities are near zero). In fact, ugly disposal can only be affected indirectly, through taxes on good and bad disposal, so the GE outcomes for ugly disposal are determined entirely by these indirect price effects. Specifically, in definitions of equations (4.1)-(4.7), each E_{ij} is a weighted average of *all* PE elasticities ($e_{ij} = 0.5$, $e_{ii} = -1$). The E_{ij} capture all the ways the PE responses are restricted in the GE model.

To see the analogous mapping for a different case in our sensitivity analysis, **Table 4** increases e_{UB} from 0.5 to 0.6 (and adjusts e_{BB} and e_{UU} to satisfy the adding-up constraint). The table shows that this change in e_{UB} from 0.5 to 0.6 makes the GE elasticity E_{UG} slightly negative. This result may seem odd, but again the logic is that this GE elasticity is a weighted average of positive and negative PE elasticities. Given the assumed symmetry in our initial assumptions, the disposal method that is changed to a stronger PE substitute for ugly disposal becomes a GE substitute, and the weaker PE substitute becomes a GE complement.

Table 4: PE Substitutes to GE Complements

Allen PE Elasticity	Assumed Value	Intermediate Elasticity	Calculated Value	Full GE Elasticity	Calculated Value
e_{UG}	0.5	ε_{UG}	0.011	E_{UG}	-0.053
e_{UB}	2.0	ε_{UB}	0.061	E_{UB}	0.051
e_{GB}	0.5	ε_{GB}	0.011	E_{GB}	0.048
e_{BG}	0.5	ε_{BG}	0.011	E_{BG}	0.105
e_{GG}	-1	ε_{GG}	-0.039	E_{GG}	-0.051
e_{BB}	-2.5	ε_{BB}	-0.089	E_{BB}	-0.099
e_{UU}	-2.5	ε_{UU}	-0.089	E_{TG}	0.486
				E_{TB}	0.461

7. Numerical Results

Using primary parameters where all cross-price elasticities are 0.5, Panel A of **Table 5** shows that the calculated value of τ_G^* is 0.061, and τ_B^* is 0.161. Both rates are higher than MED, because of revenue needs. The Corlett-Hague terms are both zero, because good waste and bad waste are equal substitutes for ugly waste – so neither can be used to reduce ugly damages. The denominators in the SBO tax formulas (5.1) and (5.2) are both 2.0, so the “own damage” term is half the MED of each waste activity. Then the revenue terms ensure budget balance.

Table 5: Tax Rate Calculations

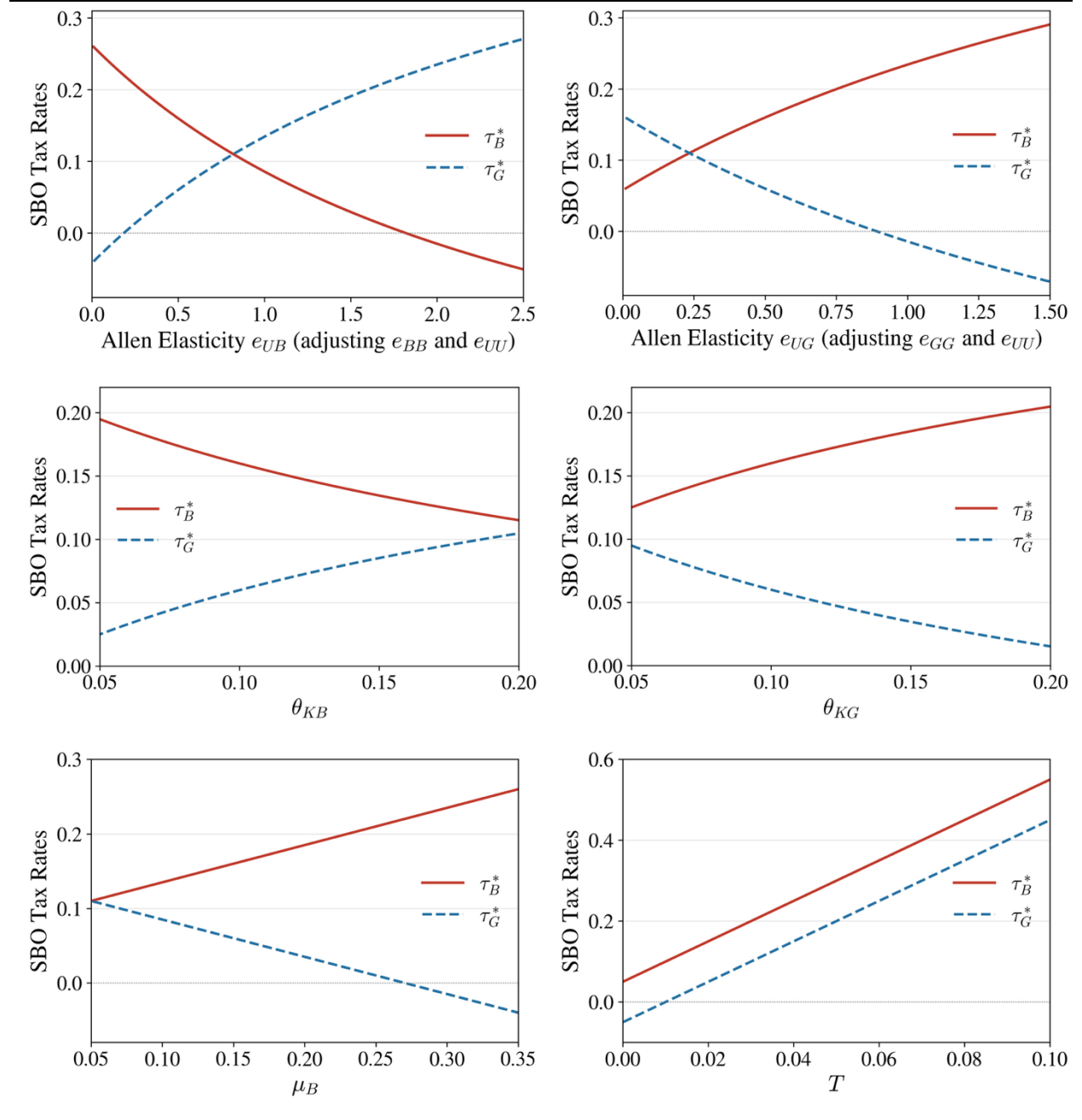
Unit Tax Rate	Components of SBO (with Denominator)				
	FBO (MED)	SBO Tax	Own Damage	Corlett-Hague Term	Revenue Term
Panel A: Using Primary Parameter Values ($e_{UB} = 0.5$)					
τ_G	0.05	0.061	0.025	0.0	0.036
τ_B	0.15	0.161	0.075	0.0	0.086
Panel B: Using Alternative Parameter Values ($e_{UB} = 2.0$)					
τ_G	0.05	0.237	0.016	0.172	0.049
τ_B	0.15	-0.014	0.101	-0.172	0.056

The FBO rates in the second column include $\tau_U = \mu_U = 0.50$. The SBO rates in the next column apply when $\tau_U = 0$. Components of SBO rates are three terms in numerators of equations (5.1) and (5.2), each divided by the denominator (so the three components add to the SBO tax rate).

Panel B uses the alternative parameters with $e_{UB} = 2.0$, which means that reducing τ_B can substantially reduce ugly damages. The large negative CH term (-0.172) is offset by a positive

“own damage” term of 0.101 and a positive revenue term of 0.056, so that the overall SBO tax rate τ_B^* becomes a subsidy (-0.014). These SBO rates face the same revenue requirement, however, so τ_G^* rises to 0.237 (despite MED of only 0.05).

Figure 1: Effects of Varying Key Parameters on the Two SBO Tax Rates



To show the effects of parameters on the two SBO tax rates, **Figure 1** varies one key parameter at a time. If U and B are better substitutes (higher e_{UB}), we expect a lower SBO tax on B . As expected, the **upper left** figure shows how a larger e_{UB} leads to a smaller τ_B^* . If e_{UB} is

high enough, τ_B^* can be less than τ_G^* or even negative. Even when landfill (or flaring of methane) has substantial external costs, for example, a tax on it encourages enough ugly pollution that the SBO policy is a subsidy to bad pollution. The figure also shows effects of a revenue requirement: a lower τ_B^* requires a higher tax on good disposal τ_G^* .

The **upper right** figure shows analogous results for e_{UG} . When U and G are weaker substitutes (leftward on the horizontal axis), the extra tax on bad disposal becomes smaller. In the other direction, if e_{UG} is high enough, then τ_G^* turns negative. The SBO policy toward good disposal can be a subsidy, in order to get all revenue from bad disposal and avoid ugly disposal.

Primary parameters start with equal weights on the three wastes ($\theta_{KG} = \theta_{KB} = \theta_{KU} = 0.1$). The **center left** of Figure 1 varies the weight on bad disposal θ_{KB} from half to twice that value (from 0.05 to 0.20). All other shares are changed proportionately so that all shares add to one.¹³ Perhaps surprisingly, the figure shows that a larger bad disposal share reduces the SBO tax on bad disposal. The explanation is that a higher share for bad disposal means more revenue from any particular tax rate, so a lower τ_B^* raises more revenue than otherwise. Analogously, the **center right** figure shows that a larger share for good disposal θ_{KG} reduces the SBO tax on good disposal. Then, given fixed total revenue, a lower τ_G^* implies a higher τ_B^* .

Next, we vary the assumed marginal external damages. Initial values are $\mu_G = 0.05$, $\mu_B = 0.15$, and $\mu_U = 0.5$. Intuitively, one might expect a higher μ_U to reduce the SBO tax rate on whichever alternative is a better substitute for ugly disposal. When other parameters remain at their initial values, however, the two taxable alternatives are equally good substitutes for ugly disposal – and the CH terms in Equations (5.1) and (5.2) reduce to zero. Then E_{UG} and E_{UB} in Table 3 equal zero. In that case, neither SBO tax rate varies with μ_U , but instead τ_B^* is always higher than τ_G^* (because μ_B is higher than μ_G). When the social planner cannot act directly *or indirectly* to reduce costly ugly disposal, the next best step is simply to control damages from bad disposal. We do not show the figure with two flat lines as μ_U varies.

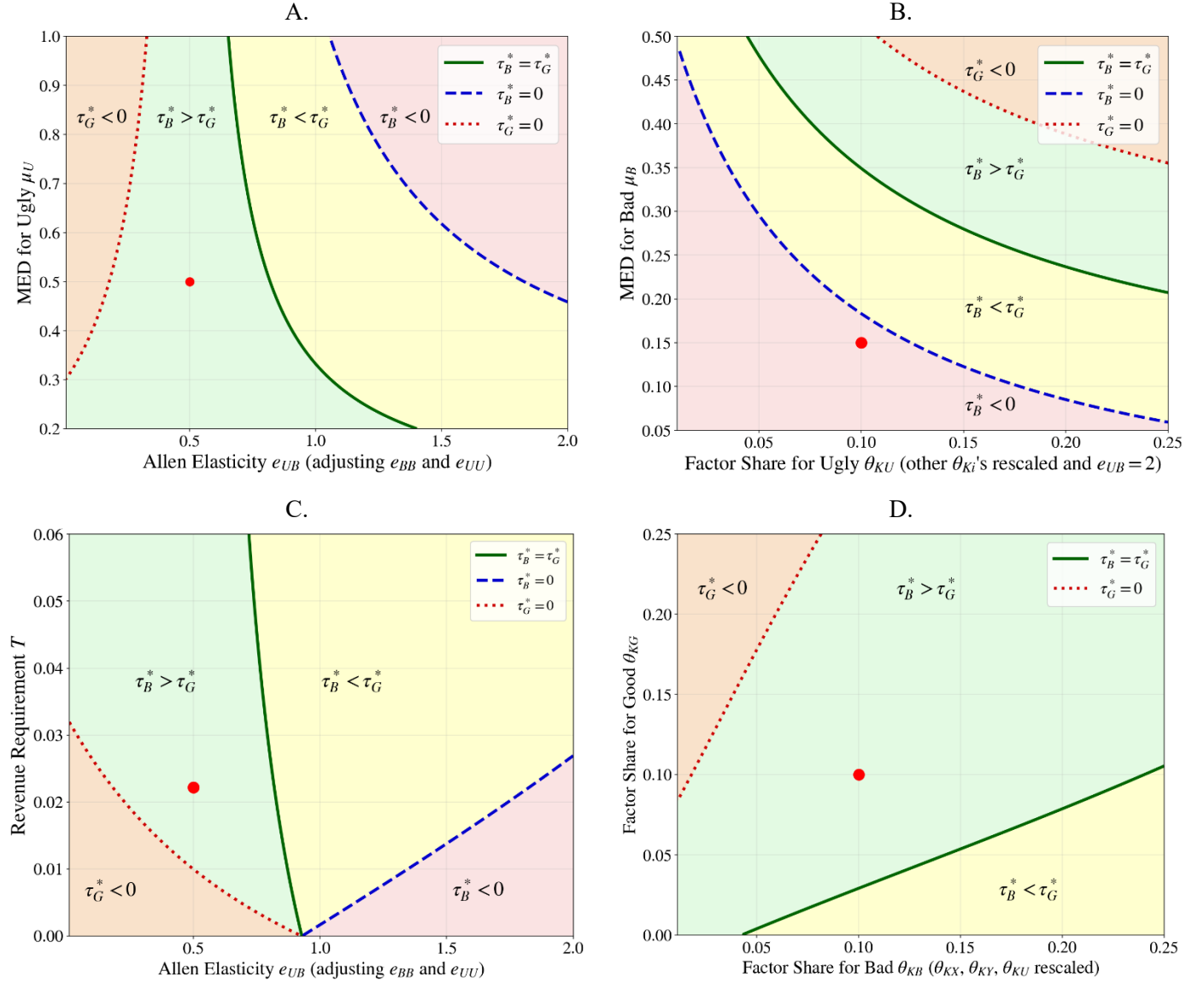
Instead, the **bottom left** of Figure 1 increases μ_B and shows that τ_B^* rises. Then, because of the fixed revenue requirement, τ_G^* can become negative with a higher μ_B .

Table 2 shows that $T = 0.022$ with all assumed primary parameter values, but the **lower right** corner of Figure 1 shows that increases in this fixed revenue requirement raises both SBO

¹³ For example, increasing θ_{KB} from 0.10 to 0.19 means that we reduce both θ_{KG} and θ_{KU} by 10% (from 0.35 to 0.315) and both θ_{KG} and θ_{KU} by 10% (from 0.10 to 0.09).

tax rates. At a revenue requirement of zero, the curve for τ_B^* starts out positive and τ_G^* is negative (to capture the fact that bad disposal is more damaging than good disposal). The difference remains constant, as both tax rates rise with the revenue requirement.

Figure 2: Relative SBO Tax Rates for Varying MED and Allen Elasticities



Next, **Figure 2** varies two parameters within each diagram. **Appendix L** provides the replication code for this figure. In the upper left panel (2A), the horizontal axis varies e_{UB} (0.5 in our primary parameters), and the vertical axis varies the MED per unit of ugly disposal, μ_U (also 0.5). The red dot shows the point with all primary parameters, where the SBO tax on bad waste exceeds the SBO tax on good waste, as expected. If we keep μ_U fixed at 0.5 and move west,

reducing e_{UB} to 0.18 or below makes τ_G^* turn negative (a subsidy to good disposal). If we instead move east, increasing e_{UB} to 0.82 or above raises the SBO tax on good disposal above the SBO tax on bad disposal. Finally, still keeping μ_U at 0.5, then increasing e_{UB} to 1.83 or above turns τ_B^* negative. In general, τ_B^* is more likely negative with a high e_{UB} , a high μ_U , or both.

In other words, the optimal policy could subsidize badly polluting behavior if the even-worse ugly disposal is highly damaging and cannot be reduced directly but can be reduced enough by subsidizing a less-polluting alternative.

As shown at the right edge of 2A where $e_{UB} = 2.0$, primary values for other parameters implies $\tau_B^* < 0$. Next, to focus on that case where bad waste is the best substitute for ugly waste, the entire panel 2B uses $e_{UB} = 2.0$. In panel 2B, the horizontal axis varies the sheer size of ugly disposal θ_{KU} (the fraction of total productive factors used for dumping, where 0.10 is the primary choice). The vertical axis varies μ_B , the MED from bad disposal. The red dot shows the point with parameters other than e_{UB} set to primary levels (*e.g.*, $\mu_B = 0.15$).

Because $e_{UB} = 2.0$ leads to a negative τ_B^* in panel 2A, the purpose of panel 2B is to show that τ_B^* is raised back above zero by increasing the MED from bad disposal (moving north) or by increasing the sheer size of ugly disposal (moving east). Thus, even when panel 2A shows that a high e_{UB} means subsidizing bad disposal (as expected by the CH effect), the northeast corner of panel 2B shows that a large weight on ugly disposal (θ_{KU}) and a high MED from bad disposal (μ_B) make it important to tax bad disposal and subsidize good disposal.

The lower left panel 2C reverts to varying e_{UB} on the horizontal axis (as in 2A), but the vertical axis varies the total revenue required (T). At low total revenue, either good disposal is subsidized if it is a better substitute for ugly (low e_{UB}), or bad disposal is subsidized if bad is a better substitute for ugly. With higher needs for revenue mean that neither is subsidized, but the better substitute for ugly still receives a lower SBO tax rate.

Finally, panel 2D varies θ_{KB} on the horizontal axis and θ_{KG} on the vertical axis. Primary parameters still yield $\tau_B^* < \tau_G^*$, but a larger share of factors used for good disposal can reduce τ_G^* below zero, while more factors for bad disposal make it easier to reduce τ_B^* below τ_G^* .

Our analytical model is general enough to study any of the six examples of pollution policy listed in our introduction – and other pollution problems as well. Any example is represented in our AGE model by a particular set of share and elasticity parameters. Thus, an expert on any specific policy problem can best choose the specific parameter values for that

problem and can find the point in our figures that best represents that policy problem – to see whether bad pollution would optimally be taxed or subsidized. For more precision, a simple program can be written using equations (5.1) and (5.2) to calculate the SBO tax rates on good and bad disposal options. Or, upon request, we authors can provide our Python program.

8. Conclusions

This paper builds and employs a new analytical general equilibrium model to find a closed-form solution for the second-best optimal tax on a relatively “good” clean production or disposal method, and the tax on a “bad” polluting production or disposal method, in the case where no tax can be imposed or enforced on the most polluting “ugly” production method (*e.g.* venting methane) or disposal method (*e.g.*, illegal dumping). This model can be applied to any environmental pollution problem, including disposal into the air, water, or land. We show that the tax on bad disposal might be *lower* than the tax on good disposal, if ugly disposal is enough better as a substitute for bad disposal than it is for good disposal. Our model captures not only differences in how PE cross-price elasticities help determine GE effects of each disposal tax, but also how all own-price and cross-price elasticities and share parameters help determine those GE outcomes, including the amount of untaxed ugly disposal. And the GE change in disposal amounts are what matter for setting the two SBO tax rates. We also show how these elasticities combine with each disposal method’s marginal external damage parameter to determine the two SBO tax rates on good and bad disposal.

We provide a formal analysis of the policy problem with untaxed pollution, but our intent is not for policymakers to insert numbers into our formulas to set actual tax rates. The analysis is intended only to clarify conceptual issues for thinking about policy. Available data and estimates are not sufficiently reliable to calibrate this model precisely for a particular problem with three types of pollution, because it would require three own-price elasticities, three cross-price elasticities, and multiple share parameters. Illegal emissions are very difficult to observe. But our model provides clear intuition about the nature of second-best policy for any pollution problem, and our results show which parameters need to be estimated to improve policymaking when some of the worst emissions cannot be directly controlled.

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The Good, the Bad and the Ugly: List of Appendices

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Appendix A – Simplified Model Welfare Analysis and Demands

This appendix has derivations used in both special cases, for welfare and for demands. For both simplified models, income is the sum of exogenous income $\bar{I}$ and transfer of revenue back to households: $I = \bar{I} + T$. With no taxes, exogenous income is enough to pay for all three goods in utility, so $\bar{I} = c_G G + c_B B + c_U U$. With taxes, the revenue is $T \equiv \tau_G G + \tau_B B + \tau_U U$. Households get back the revenue per household, T , but they view it as fixed. They maximize:

$$\mathcal{L} = \Psi[G, B, U; nG, nB, nU] + \lambda[I - p_G G - p_B B - p_U U] \quad (1.1)$$

Step 1A. When households choose quantities of G , B , and U to maximize utility subject to their budget constraint, they take as given all income $I = \bar{I} + T$ and externalities nG , nB , and nU .

First-order conditions (FOC) are:

$$\frac{\partial \Psi}{\partial G} = \lambda p_G, \quad \frac{\partial \Psi}{\partial B} = \lambda p_B, \quad \text{and} \quad \frac{\partial \Psi}{\partial U} = \lambda p_U.$$

The omniscient social planner knows those FOC and knows that any tax shock can change each of the quantities in the household choices, in the externalities, and in the revenue transfer, T . For the social planner, only $\bar{I}$ is exogenous.

Step 2A. The social planner totally differentiates utility and inserts first-order conditions to get:

$$d\Psi = \lambda p_G dG + \lambda p_B dB + \lambda p_U dU + \frac{\partial \Psi}{\partial nG} n dG + \frac{\partial \Psi}{\partial nB} n dB + \frac{\partial \Psi}{\partial nU} n dU.$$

Step 3A. Divide by λ , the marginal utility of income, to convert $d\Psi$ into the dollar value of the change in utility ($\frac{d\Psi}{\lambda}$). Then use the definitions for the marginal damage per unit of disposal:

$\mu_G \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial n_G}$, $\mu_B \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial n_B}$, and $\mu_U \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial n_U}$ to get:

$$\frac{d\Psi}{\lambda} = (p_G - \mu_G)dG + (p_B - \mu_B)dB + (p_U - \mu_U)dU.$$

Step 4A. The price per unit of disposal is the sum of a cost and potential tax ($p_i = c_i + \tau_i$ for $i = G, B, U$). Substitute the per unit cost and per unit tax for prices to get:

$$\frac{d\Psi}{\lambda} = (\tau_G - \mu_G)dG + (\tau_B - \mu_B)dB + (\tau_U - \mu_U)dU + c_G dG + c_B dB + c_U dU.$$

Step 5A. The household budget constraint is: $I = \bar{I} + T = p_G G + p_B B + p_U U$. In that budget, $\bar{I}$ is $c_G G + c_B B + c_U U$. By definition of exogenous income, we have $d\bar{I} = 0$. Therefore, $d\bar{I} = c_G dG + c_B dB + c_U dU = 0$, and the equation in the previous step (4A) becomes:

$$\frac{d\Psi}{\lambda} = (\tau_G - \mu_G)dG + (\tau_B - \mu_B)dB + (\tau_U - \mu_U)dU.$$

Step 6A. Divide both sides by income I , to express that cash-equivalent change as a fraction of income. Then use the hat definitions (e.g., $\hat{G} \equiv dG/G$) to get:

$$\frac{d\Psi}{\lambda I} = (\tau_G - \mu_G) \frac{G}{I} \hat{G} + (\tau_B - \mu_B) \frac{B}{I} \hat{B} + (\tau_U - \mu_U) \frac{U}{I} \hat{U} \quad (1.2)$$

Step 7A. Next, we linearize the demand equations. Equations (2.1)–(2.3) are the demand functions for disposal: $G = G(p_G, p_B, p_U, I)$, $B = B(p_G, p_B, p_U, I)$, and $U = U(p_G, p_B, p_U, I)$. These three demand functions are strictly analogous to each other, so we show the derivation only for G . First, totally differentiate $G = G(p_G, p_B, p_U, I)$ to get:

$$dG = \frac{\partial G}{\partial p_G} dp_G + \frac{\partial G}{\partial p_B} dp_B + \frac{\partial G}{\partial p_U} dp_U + \frac{\partial G}{\partial I} dI.$$

Then divide by G , and multiply each term on the right by one (e.g., $\frac{p_G}{p_G}$, $\frac{p_B}{p_B}$, $\frac{p_U}{p_U}$, or $\frac{I}{I}$) to get:

$$\frac{dG}{G} = \frac{\partial G}{\partial p_G} \frac{p_G}{G} \frac{dp_G}{p_G} + \frac{\partial G}{\partial p_B} \frac{p_B}{G} \frac{dp_B}{p_B} + \frac{\partial G}{\partial p_U} \frac{p_U}{G} \frac{dp_U}{p_U} + \frac{\partial G}{\partial I} \frac{I}{G} \frac{dI}{I}.$$

Use hat definitions on G and the prices to get:

$$\hat{G} = \frac{\partial G}{\partial p_G} \frac{p_G}{G} \hat{p}_G + \frac{\partial G}{\partial p_B} \frac{p_B}{G} \hat{p}_B + \frac{\partial G}{\partial p_U} \frac{p_U}{G} \hat{p}_U + \frac{\partial G}{\partial I} \frac{I}{G} \hat{I}.$$

Assume homothetic preferences among all three disposal methods, which means that the elasticity of demand with respect to income must be one for each disposal method ($\frac{\partial G}{\partial I} \frac{I}{G} =$

$\frac{\partial B}{\partial I} \frac{I}{B} = \frac{\partial U}{\partial I} \frac{I}{U} = 1$). Then we can write the entire equation as:

$$\hat{G} = a_{GG}\hat{p}_G + a_{GB}\hat{p}_B + a_{GU}\hat{p}_U + \hat{I}, \quad (2.1)$$

where $a_{ij} \equiv \frac{\partial i}{\partial p_j} \frac{p_j}{i}$ (for $i, j = G, B, U$). In general, each a_{ij} is the elasticity of demand for i with respect to a change in the price of j . Following the same logic explained in greater detail in Appendix D, we use the properties of the Allen elasticities that says $a_{ij} = e_{ij}\theta_{ij}$, where e_{ij} is the Allen partial elasticity of substitution and $\theta_{ij} \equiv \frac{p_j}{I}$ is the share of income spent on j . Allen also shows that $e_{ii} < 0$, that $e_{ij} = e_{ji}$, and that $a_{iG} + a_{iB} + a_{iU} = 0$.

Step 8A. Next, we linearize equations for changes in prices. In equations (2.4)–(2.6), the price per unit of disposal p_i is the sum of a cost per unit c_i and potential tax per unit τ_i ($p_i = c_i + \tau_i$ for $i = G, B, U$). Totally differentiate $p_G = c_G + \tau_G$:

$$dp_G = dc_G + d\tau_G.$$

The cost per unit is fixed ($dc_G = 0$). Divide both sides by p_G , and multiply $d\tau_G$ by one ($\frac{\tau_G}{\tau_G}$):

$$\frac{dp_G}{p_G} = \frac{\tau_G}{p_G} \frac{d\tau_G}{\tau_G}.$$

Use the hat definitions and the *ad valorem* rate for good disposal to get:

$$\hat{p}_G = t_G \hat{\tau}_G, \quad (2.4)$$

where $t_G \equiv \tau_G/p_G$ is the definition for the *ad valorem* tax rate for good disposal. Derivations for the changes in prices of bad and ugly disposal are analogous: $\hat{p}_B = t_B \hat{\tau}_B$ and $\hat{p}_U = t_U \hat{\tau}_U$. When ugly disposal cannot be taxed, $\tau_U = 0$, so the price of ugly doesn't change ($\hat{p}_U = 0$).

Appendix B – First Special Case: Corlett & Hague (No Externalities, No Lump Sum Tax)

Here we derive equation (2.8) in the main text, the ratio of *ad valorem* tax rates with no externalities and no ability to impose a lump-sum tax. First, we linearize the revenue equation for taxes on good and bad disposal and substitute all linearized equations into the welfare equation (1.2). The solution below matches equation (3.21) in Auerbach and Hines (2002).

For Corlett and Hague, exogenous income is fixed and the transfer T is fixed, so income is fixed ($\hat{I} = 0$). The linearized demand equations (2.1) to (2.3) are then: $\hat{G} = a_{GG}\hat{p}_G + a_{GB}\hat{p}_B + a_{GU}\hat{p}_U$, $\hat{B} = a_{BG}\hat{p}_G + a_{BB}\hat{p}_B + a_{BU}\hat{p}_U$, and $\hat{U} = a_{UG}\hat{p}_G + a_{UB}\hat{p}_B + a_{UU}\hat{p}_U$.

Step 1B. We now linearize the government revenue equation to describe how an exogenous change in one tax rate is met with an endogenous change to other. Equation (2.7) for tax revenue says $T = \tau_G G + \tau_B B$. Total differentiation yields $dT = d\tau_G G + \tau_G dG + d\tau_B B + \tau_B dB$. Divide

by T to get: $\frac{dT}{T} = \frac{d\tau_G G}{T} + \frac{dG \tau_G}{T} + \frac{d\tau_B B}{T} + \frac{dB \tau_B}{T}$. Multiply each term on the right by one (e.g., $\frac{\tau_G}{\tau_G}$, $\frac{G}{G}$, $\frac{\tau_B}{\tau_B}$, or $\frac{B}{B}$) to get:

$$\hat{T} = \frac{d\tau_G}{\tau_G} \frac{\tau_G G}{T} + \frac{dG}{G} \frac{G \tau_G}{T} + \frac{d\tau_B}{\tau_B} \frac{\tau_B B}{T} + \frac{dB}{B} \frac{B \tau_B}{T}.$$

Use the hat definitions to get:

$$\hat{T} = \theta_{TG}(\widehat{\tau}_G + \hat{G}) + \theta_{TB}(\widehat{\tau}_B + \hat{B}), \quad (2.7)$$

where $\theta_{TG} \equiv \frac{\tau_G G}{T}$ and $\theta_{TB} \equiv \frac{\tau_B B}{T}$. With a fixed government budget, any increase in one tax rate results in a decrease to the other tax rate (on the normal side of its Laffer curve). Solve for $\widehat{\tau}_G$ in terms of $\widehat{\tau}_B$ by first setting $\hat{T} = 0$ to get:

$$0 = \theta_{TG} \widehat{\tau}_G + \theta_{TB} \widehat{\tau}_B + \theta_{TG} \hat{G} + \theta_{TB} \hat{B}.$$

Step 2B. For the Corlett and Hague model with no externalities and no policy variable on ugly, set $\mu_G = \mu_B = \mu_U = 0$ and $\tau_U = 0$. At the optimal rates, no change in either tax rate raises welfare ($d\Psi = 0$). The welfare equation (1.2) becomes:

$$0 = \tau_G \frac{G}{I} \hat{G} + \tau_B \frac{B}{I} \hat{B}.$$

Step 3B. To solve for $\widehat{\tau}_G$ in terms of $\widehat{\tau}_B$, multiply both sides of the welfare equation from Step 2B by $\frac{I}{T}$ and use the definitions of the revenue shares $\theta_{TG} \equiv \frac{\tau_G G}{T}$ and $\theta_{TB} \equiv \frac{\tau_B B}{T}$ to get: $0 = \theta_{TG} \hat{G} + \theta_{TB} \hat{B}$. Set the Corlett and Hague welfare equation equal to linearized revenue equation:

$$\theta_{TG} \hat{G} + \theta_{TB} \hat{B} = \theta_{TG} \hat{G} + \theta_{TB} \hat{B} + \theta_{TG} \widehat{\tau}_G + \theta_{TB} \widehat{\tau}_B = 0.$$

Then subtract $\theta_{TG} \hat{G} + \theta_{TB} \hat{B} = 0$ from government budget equation to get:

$$0 = \theta_{TG} \widehat{\tau}_G + \theta_{TB} \widehat{\tau}_B.$$

Subtract $\theta_{TB} \widehat{\tau}_B$ from both sides, then divide by θ_{TG} to get $\widehat{\tau}_G$ in terms of $\widehat{\tau}_B$:

$$\widehat{\tau}_G = -\frac{\theta_{TB}}{\theta_{TG}} \widehat{\tau}_B.$$

Step 4B. Substitute linear equations for $\hat{G}$ and $\hat{B}$ into the resulting equation from Step 2B. Then substitute linear equations for $\widehat{p}_G$ and $\widehat{p}_B$ to get:

$$0 = \tau_G \frac{G}{I} (a_{GG} t_G \widehat{\tau}_G + a_{GB} t_B \widehat{\tau}_B) + \tau_B \frac{B}{I} (a_{BG} t_G \widehat{\tau}_G + a_{BB} t_B \widehat{\tau}_B).$$

Step 5B. Substitute the equation $\widehat{\tau}_G = -\frac{\theta_{TB}}{\theta_{TG}} \widehat{\tau}_B$ from Step 3B, factor out $\widehat{\tau}_B$ from both brackets, and divide both sides by $\widehat{\tau}_B$ to get:

$$0 = \tau_G \frac{G}{I} \left(a_{GB} t_B - a_{GG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right) + \tau_B \frac{B}{I} \left(a_{BB} t_B - a_{BG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right).$$

Step 6B. Rewrite τ_G and τ_B in terms of their prices and *ad valorem* rates. Substitute $t_G p_G$ for τ_G and $t_B p_B$ for τ_B to get:

$$0 = \frac{p_G G}{I} \left(a_{GB} t_B - a_{GG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right) t_G + \frac{p_B B}{I} \left(a_{BB} t_B - a_{BG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right) t_B.$$

Step 7B. Substitute θ_{IG} , the share of income spent on disposal G , for $\frac{p_G G}{I}$ and θ_{IB} for $\frac{p_B B}{I}$ to get:

$$0 = \theta_{IG} \left(a_{GB} t_B - a_{GG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right) t_G + \theta_{IB} \left(a_{BB} t_B - a_{BG} t_G \frac{\theta_{TB}}{\theta_{TG}} \right) t_B.$$

Step 8B. The definition of the government revenue shares allows us to write the ratio $\frac{\theta_{TB}}{\theta_{TG}}$ as

$\frac{t_B p_B B / T}{t_G p_G G / T}$. Cancel the T in the numerator and denominator to get $\frac{t_B p_B B}{t_G p_G G}$. Then multiply by one $(\frac{1/I}{1/I})$

to get $\frac{t_B p_B B / I}{t_G p_G G / I}$. The definitions for the share of income spent on G and B allow us to rewrite this

ratio as $\frac{\theta_{IB} t_B}{\theta_{IG} t_G}$. Substitute $\frac{\theta_{IB} t_B}{\theta_{IG} t_G}$ for $\frac{\theta_{TB}}{\theta_{TG}}$ to get:

$$0 = \theta_{IG} \left(a_{GB} t_B - a_{GG} t_G \frac{\theta_{IB} t_B}{\theta_{IG} t_G} \right) t_G + \theta_{IB} \left(a_{BB} t_B - a_{BG} t_G \frac{\theta_{IB} t_B}{\theta_{IG} t_G} \right) t_B.$$

Step 9B. Substitute $e_{ij} \theta_{Ij}$ for each a_{ij} to get:

$$0 = \theta_{IG} \left(\theta_{IB} e_{GB} t_B - \theta_{IG} e_{GG} t_G \frac{\theta_{IB} t_B}{\theta_{IG} t_G} \right) t_G + \theta_{IB} \left(\theta_{IB} e_{BB} t_B - \theta_{IG} e_{BG} t_G \frac{\theta_{IB} t_B}{\theta_{IG} t_G} \right) t_B.$$

Step 10B. In the second term of each bracket, cancel $\theta_{IG} t_G$ found in the numerator and denominator. Then expand the brackets and divide both sides $\theta_{IB} t_B$ to get:

$$0 = \theta_{IG} e_{GB} t_G - \theta_{IG} e_{GG} t_G + \theta_{IB} e_{BB} t_B - e_{BG} \theta_{IB} t_B.$$

Step 11B. Add $e_{BG} \theta_{IB} t_B$ and subtract $\theta_{IB} e_{BB} t_B$ from both sides:

$$e_{BG} \theta_{IB} t_B - \theta_{IB} e_{BB} t_B = \theta_{IG} e_{GB} t_G - \theta_{IG} e_{GG} t_G.$$

Step 12B. The symmetry properties of Allen elasticities allow us to rewrite e_{BG} as e_{GB} on the left side of the equation and e_{GB} as e_{BG} on the right side of the equation. Substitute a_{ij} for each $e_{ij} \theta_{Ij}$ to get:

$$a_{GB} t_B - a_{BB} t_B = a_{BG} t_G - a_{GG} t_G.$$

Step 13B. Factor out t_B on the LHS and t_G on the RHS. Then divide both sides by t_G and $(a_{GB} - a_{BB})$ to get:

$$\frac{t_B}{t_G} = \frac{(a_{BG} - a_{GG})}{(a_{GB} - a_{BB})}.$$

Step 14B. The adding-up constraint says $a_{iG} + a_{iB} + a_{iU} = 0$. This equation allows us to substitute $a_{GG} = -a_{GB} - a_{GU}$ and $a_{BB} = -a_{BG} - a_{BU}$ to get:

$$\frac{t_B}{t_G} = \frac{a_{GU} + a_{GB} + a_{BG}}{a_{BU} + a_{GB} + a_{BG}} \quad (2.8)$$

Appendix C – Second Special Case: Sallee (with Externalities, with Lump Sum Tax)

This Appendix derives an equation equivalent to Sallee (2025) with only three polluting goods when policy variables exist on just two of the polluting goods. We also give the planner the ability to raise revenue through a lump-sum tax. Sallee's model includes two clean goods, two dirty goods, and a numeraire.

Step 1C. Welfare is maximized when $d\Psi = 0$. With no tax on ugly, set $\tau_U = 0$ in (1.2) to get:

$$0 = (\tau_G - \mu_G) \frac{G}{I} \hat{G} + (\tau_B - \mu_B) \frac{B}{I} \hat{B} - \mu_U \frac{U}{I} \hat{U}.$$

Step 2C. Substitute equations (2.1)–(2.3) for $\hat{G}$, $\hat{B}$, and $\hat{U}$ with $\hat{p}_U = 0$, and then multiply both sides by I to get:

$$0 = (\tau_G - \mu_G)G(a_{GG}\hat{p}_G + a_{GB}\hat{p}_B) + (\tau_B - \mu_B)B(a_{BG}\hat{p}_G + a_{BB}\hat{p}_B) - \mu_U U(a_{UG}\hat{p}_G + a_{UB}\hat{p}_B).$$

Step 3C. Substitute equations (2.4) and (2.5) for $\hat{p}_G$ and $\hat{p}_B$ to get:

$$0 = (\tau_G - \mu_G)G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B) + (\tau_B - \mu_B)B(a_{BG}t_G\hat{\tau}_G + a_{BB}t_B\hat{\tau}_B) - \mu_U U(a_{UG}t_G\hat{\tau}_G + a_{UB}t_B\hat{\tau}_B).$$

Step 4C. Subtract $\tau_G G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B)$ from both sides to get:

$$-\tau_G G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B) = -\mu_G G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B) + (\tau_B - \mu_B)B(a_{BG}t_G\hat{\tau}_G + a_{BB}t_B\hat{\tau}_B) - \mu_U U(a_{UG}t_G\hat{\tau}_G + a_{UB}t_B\hat{\tau}_B).$$

Step 5C. Divide both sides by $-G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B)$ to get:

$$\tau_G = \mu_G + \mu_U \frac{U(a_{UG}t_G\hat{\tau}_G + a_{UB}t_B\hat{\tau}_B)}{G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B)} + (\mu_B - \tau_B) \frac{B(a_{BG}t_G\hat{\tau}_G + a_{BB}t_B\hat{\tau}_B)}{G(a_{GG}t_G\hat{\tau}_G + a_{GB}t_B\hat{\tau}_B)}.$$

Step 6C. Sallee (2025) considers a subsidy-first approach with two clean goods and two dirty goods, when the tax on both dirty goods is zero while our paper only considers only one untaxed dirty good. Consider the case where $\mu_B = \tau_B$, which implies $\hat{\tau}_B = 0$, and the planner only chooses only τ_G^* . The equation above then can then be written:

$$\tau_G^* = \mu_G + \mu_U \frac{U a_{UG} t_G \widehat{\tau}_G}{G a_{GG} t_G \widehat{\tau}_G}.$$

Step 7C. Cancel the terms $t_G \widehat{\tau}_G$ in numerator and denominator to get:

$$\tau_G^* = \mu_G + \mu_U \frac{U a_{UG}}{G a_{GG}}.$$

Step 8C. The elasticities in the equation above are defined as $a_{UG} \equiv \frac{\partial U}{\partial p_G} \frac{p_G}{U}$ and $a_{GG} \equiv \frac{\partial G}{\partial p_G} \frac{p_G}{G}$.

Since c_G is fixed and $p_G = c_G + \tau_G$, a change in p_G is equal to a change in τ_G . Therefore, the derivative of any quantity with respect to p_G is equal to its derivative with respect to τ_G .

Substitute $\frac{\partial U}{\partial \tau_G} \frac{p_G}{U}$ for a_{UG} and $\frac{\partial G}{\partial \tau_G} \frac{p_G}{G}$ for a_{GG} to get:

$$\tau_G = \mu_G + \mu_U \frac{U \frac{\partial U}{\partial \tau_G} \frac{p_G}{U}}{G \frac{\partial G}{\partial \tau_G} \frac{p_G}{G}}$$

Step 9C. Cancel U , G , and p_G to get:

$$\tau_G = \mu_G + \mu_U \frac{\partial U / \partial \tau_G}{\partial G / \partial \tau_G} = \mu_G - \mu_U \frac{-\partial U / \partial \tau_G}{\partial G / \partial \tau_G}$$

This is equivalent to equation (5) in Saltee (2025) shown below with no second untaxed dirty good ($\phi_4 = 0$).

$$\tilde{s}_1 = \phi_1 - \phi_3 \frac{-\partial D_3 / \partial s_1}{\partial G_1 / \partial s_1} - \phi_4 \frac{-\partial D_4 / \partial s_1}{\partial G_1 / \partial s_1}. \quad (5)$$

Derive simultaneous closed-form solutions for τ_B^* and τ_G^*

Step 10C. Conduct two policy experiments: First, $\widehat{\tau}_G > 0$ and $\widehat{\tau}_B = 0$, and then second, consider $\widehat{\tau}_B > 0$ and $\widehat{\tau}_G = 0$. The first experiment yields the following equation:

$$\tau_G = \mu_G + \mu_U \frac{U(a_{UG} t_G \widehat{\tau}_G)}{G(a_{GG} t_G \widehat{\tau}_G)} + (\mu_B - \tau_B) \frac{B(a_{BG} t_G \widehat{\tau}_G)}{G(a_{GG} t_G \widehat{\tau}_G)}.$$

The second experiment yields the equation:

$$\tau_G = \mu_G + \mu_U \frac{U(a_{UB} t_B \widehat{\tau}_B)}{G(a_{GB} t_B \widehat{\tau}_B)} + (\mu_B - \tau_B) \frac{B(a_{BB} t_B \widehat{\tau}_B)}{G(a_{GB} t_B \widehat{\tau}_B)}.$$

Step 11C. Cancel out $t_G \widehat{\tau}_G$ in the numerator and denominator of the first experiment to get:

$$\tau_G = \mu_G + \mu_U \frac{U a_{UG}}{G a_{GG}} + (\mu_B - \tau_B) \frac{B a_{BG}}{G a_{GG}}.$$

Cancel out $t_B \widehat{\tau}_B$ in the numerator and denominator of the second experiment to get:

$$\tau_G = \mu_G + \mu_U \frac{U a_{UB}}{G a_{GB}} + (\mu_B - \tau_B) \frac{B a_{BB}}{G a_{GB}}.$$

Step 12C. We now have two equations for τ_G from two policy experiments. Set equations for τ_G equal to each other, and then solve for τ_B^* :

$$\mu_G + \mu_U \frac{U a_{UG}}{G a_{GG}} + (\mu_B - \tau_B^*) \frac{B a_{BG}}{G a_{GG}} = \mu_G + \mu_U \frac{U a_{UB}}{G a_{GB}} + (\mu_B - \tau_B^*) \frac{B a_{BB}}{G a_{GB}}$$

Step 13C. Subtract μ_G , $\mu_U \frac{U a_{UG}}{G a_{GG}}$, and $\mu_B \frac{B a_{BG}}{G a_{GG}}$ from both sides to get:

$$-\tau_B^* \frac{B a_{BG}}{G a_{GG}} = \mu_U \frac{U a_{UB}}{G a_{GB}} - \mu_U \frac{U a_{UG}}{G a_{GG}} + (\mu_B - \tau_B^*) \frac{B a_{BB}}{G a_{GB}} - \mu_B \frac{B a_{BG}}{G a_{GG}}$$

Step 14C. Add $\tau_B^* \frac{B a_{BB}}{G a_{GB}}$ to both sides to get:

$$\tau_B^* \frac{B a_{BB}}{G a_{GB}} - \tau_B^* \frac{B a_{BG}}{G a_{GG}} = \mu_U \frac{U a_{UB}}{G a_{GB}} - \mu_U \frac{U a_{UG}}{G a_{GG}} + \mu_B \frac{B a_{BB}}{G a_{GB}} - \mu_B \frac{B a_{BG}}{G a_{GG}}$$

Step 15C. Multiply each term by one (*e.g.*, $\frac{a_{GG}}{a_{GB}}$ or $\frac{a_{GB}}{a_{GB}}$), then combine like-terms to get:

$$\tau_B^* \frac{B(a_{BB}a_{GG} - a_{BG}a_{GB})}{G a_{GB}a_{GG}} = \mu_U \frac{U(a_{UB}a_{GG} - a_{UG}a_{GB})}{G a_{GB}a_{GG}} + \mu_B \frac{B(a_{BB}a_{GG} - a_{BG}a_{GB})}{G a_{GB}a_{GG}}$$

Step 16C. Multiply both sides by $G a_{GB} a_{GG}$ to get:

$$\tau_B^* B(a_{BB}a_{GG} - a_{BG}a_{GB}) = \mu_U U(a_{UB}a_{GG} - a_{UG}a_{GB}) + \mu_B B(a_{BB}a_{GG} - a_{BG}a_{GB})$$

Step 17C. Divide both sides by $B(a_{BB}a_{GG} - a_{BG}a_{GB})$ to get τ_B^* in our text's equation (2.9b):

$$\tau_B^* = \mu_B + \mu_U \frac{U(a_{UB}a_{GG} - a_{UG}a_{GB})}{B(a_{BB}a_{GG} - a_{BG}a_{GB})} \quad (2.9b)$$

Step 18C. Steps to solve for the τ_G^* are analogous and yield:

$$\tau_G^* = \mu_G + \mu_U \frac{U(a_{UB}a_{BG} - a_{UG}a_{BB})}{G(a_{GB}a_{BG} - a_{GG}a_{BB})} \quad (2.9a)$$

Equation (2.9) mapped to Sallee's Notation.

Step 19C. Substitute the definitions of the elasticities $a_{ij} \equiv \frac{\partial i}{\partial p_j} \frac{p_j}{i}$ for i and $j = G, B, U$. Then use the same reasoning from Step 8C and substitute the derivative with respect to τ_j for the derivative with respect to p_j for $j = G, B$ to get:

$$\tau_B^* = \mu_B + \mu_U \frac{U \frac{\partial U}{\partial \tau_B} \frac{p_B}{U} \frac{\partial G}{\partial \tau_G} \frac{p_G}{G} - \frac{\partial U}{\partial \tau_G} \frac{p_G}{U} \frac{\partial G}{\partial \tau_B} \frac{p_B}{G}}{B \frac{\partial B}{\partial \tau_B} \frac{p_B}{B} \frac{\partial G}{\partial \tau_G} \frac{p_G}{G} - \frac{\partial B}{\partial \tau_G} \frac{p_G}{B} \frac{\partial G}{\partial \tau_B} \frac{p_B}{G}}.$$

Step 20C. Cancel U and B to get:

$$\tau_B^* = \mu_B + \mu_U \frac{\frac{\partial U}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} \frac{p_B p_G}{G} - \frac{\partial U}{\partial \tau_G} \frac{\partial G}{\partial \tau_B} \frac{p_B p_G}{G}}{\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} \frac{p_B p_G}{G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B} \frac{p_B p_G}{G}}.$$

Step 21C. Cancel the terms $\frac{p_B p_G}{G}$ in the numerator and denominator to get:

$$\tau_B^* = \mu_B + \mu_U \frac{\frac{\partial U}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial U}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}{\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}.$$

Step 22C. Multiply the term containing μ_U by the one $\left(\frac{\frac{1}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}}{\frac{1}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}} \right)$ to get:

$$\tau_B^* = \mu_B + \mu_U \frac{\frac{\frac{\partial U}{\partial \tau_B} \frac{\partial G}{\partial \tau_G}}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}} - \frac{\frac{\partial U}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}}{\frac{\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G}}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}} - \frac{\frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}{\frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}}}.$$

Step 23C. Cancel common factors to get:

$$\tau_B^* = \mu_B - \mu_U \frac{\frac{\frac{\partial U}{\partial \tau_G}}{\frac{\partial G}{\partial \tau_G}} - \frac{\frac{\partial U}{\partial \tau_B}}{\frac{\partial G}{\partial \tau_B}}}{\frac{\frac{\partial B}{\partial \tau_B}}{\frac{\partial G}{\partial \tau_B}} - \frac{\frac{\partial B}{\partial \tau_G}}{\frac{\partial G}{\partial \tau_G}}}.$$

Step 24C. Steps for analogous for τ_G^* :

$$\tau_G^* = \mu_G - \mu_U \frac{\frac{\frac{\partial U}{\partial \tau_G}}{\frac{\partial B}{\partial \tau_G}} - \frac{\frac{\partial U}{\partial \tau_B}}{\frac{\partial B}{\partial \tau_B}}}{\frac{\frac{\partial G}{\partial \tau_B}}{\frac{\partial B}{\partial \tau_B}} - \frac{\frac{\partial G}{\partial \tau_G}}{\frac{\partial B}{\partial \tau_G}}}.$$

Without a second untaxed dirty good ($\phi_4 = 0$), the SBO rates are equivalent to equations (3) and (4) in Sallee (2025) shown below:

$$s_1^* = \phi_1 - \phi_3 \left[\frac{\frac{\partial D_3 / \partial s_2}{\partial G_2 / \partial s_2} - \frac{\partial D_3 / \partial s_1}{\partial G_2 / \partial s_1}}{\frac{\partial G_1 / \partial s_1}{\partial G_2 / \partial s_1} - \frac{\partial G_1 / \partial s_2}{\partial G_2 / \partial s_2}} \right] - \phi_4 \left[\frac{\frac{\partial D_4 / \partial s_2}{\partial G_2 / \partial s_2} - \frac{\partial D_4 / \partial s_1}{\partial G_2 / \partial s_1}}{\frac{\partial G_1 / \partial s_1}{\partial G_2 / \partial s_1} - \frac{\partial G_1 / \partial s_2}{\partial G_2 / \partial s_2}} \right], \quad (3)$$

$$s_2^* = \phi_2 - \phi_3 \left[\frac{\frac{\partial D_3 / \partial s_1}{\partial G_1 / \partial s_1} - \frac{\partial D_3 / \partial s_2}{\partial G_1 / \partial s_2}}{\frac{\partial G_2 / \partial s_2}{\partial G_1 / \partial s_2} - \frac{\partial G_2 / \partial s_1}{\partial G_1 / \partial s_1}} \right] - \phi_4 \left[\frac{\frac{\partial D_4 / \partial s_1}{\partial G_1 / \partial s_1} - \frac{\partial D_4 / \partial s_2}{\partial G_1 / \partial s_2}}{\frac{\partial G_2 / \partial s_2}{\partial G_1 / \partial s_2} - \frac{\partial G_2 / \partial s_1}{\partial G_1 / \partial s_1}} \right]. \quad (4)$$

Proof: Equation (2.9) is the FBO solution.

The equation for τ_B^* from Step 21C says:

$$\tau_B^* = \mu_B + \mu_U \frac{\partial U / \partial \tau_B \partial G / \partial \tau_G - \partial U / \partial \tau_G \partial G / \partial \tau_B}{\partial B / \partial \tau_B \partial G / \partial \tau_G - \partial B / \partial \tau_G \partial G / \partial \tau_B}.$$

Step 25C. Take the partial derivative of the budget constraint $\bar{I} = c_G G + c_B B + c_U U$ with respect to τ_B to get $c_U \frac{\partial U}{\partial \tau_B} = -c_G \frac{\partial G}{\partial \tau_B} - c_B \frac{\partial B}{\partial \tau_B}$. When c_G , c_B , and c_U are normalized to one, the partial derivative for ugly with respect to τ_B is $\frac{\partial U}{\partial \tau_B} = -\frac{\partial G}{\partial \tau_B} - \frac{\partial B}{\partial \tau_B}$. Analogously, the partial for ugly with respect to τ_G is $\frac{\partial U}{\partial \tau_G} = -\frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G}$. Substitute $\left(-\frac{\partial G}{\partial \tau_B} - \frac{\partial B}{\partial \tau_B}\right)$ for $\frac{\partial U}{\partial \tau_B}$ and $\left(-\frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G}\right)$ for $\frac{\partial U}{\partial \tau_G}$:

$$\tau_B^* = \mu_B + \mu_U \frac{\left(-\frac{\partial G}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} + \frac{\partial G}{\partial \tau_G} \frac{\partial G}{\partial \tau_B} + \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}{\left(\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}$$

Step 26C. Cancel terms in the numerator to get:

$$\tau_B^* = \mu_B + \mu_U \frac{\left(-\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} + \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}{\left(\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}$$

Step 27C. Factor out -1 from the numerator of the μ_U term to get:

$$\tau_B^* = \mu_B - \mu_U \frac{\left(\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}{\left(\frac{\partial B}{\partial \tau_B} \frac{\partial G}{\partial \tau_G} - \frac{\partial B}{\partial \tau_G} \frac{\partial G}{\partial \tau_B}\right)}$$

Step 28C. The numerator and denominator are identical, divide to get:

$$\tau_B^* = \mu_B - \mu_U \text{ and analogously, } \tau_G^* = \mu_G - \mu_U \quad (2.10)$$

Also, writing equations (2.9) as partial derivatives yields that same result when c_G , c_B , and c_U are normalized to one.

Appendix D – Linearization of 19 Equations in the Full GE Model

Equation (3.1) says $X = K_X / c_X$. Differentiation yields $dX = \frac{dK_X}{c_X}$. Divide by $X = \frac{K_X}{c_X}$ to get $\frac{dX}{X} = \left(\frac{dK_X}{c_X}\right) / \left(\frac{K_X}{c_X}\right)$. The c_X terms cancel. Using our “hat” definitions (e.g., $\hat{X} \equiv \frac{dX}{X}$), we get $\hat{X} = \widehat{K_X}$.

Equation (3.2) says $p_X X = p_K K_X$. Total differentiation yields $dp_X X + p_X dX = dp_K K_X + p_K dK_X$. Divide all four terms by either $p_X X$ or equivalently by $p_K K_X$, and use the hat definitions to get $\widehat{p_X} + \hat{X} = \widehat{p_K} + \widehat{K_X}$.

Equations (3.3) and (3.4) for output Y are derived in exactly the same way as (3.1) and (3.2).

Equation (3.5) says $K_G = c_G G$. Differentiation yields $dK_G = c_G dG$. Divide by K_G to get $\frac{dK_G}{K_G} = \frac{c_G dG}{c_G G}$. Simplify and use the hat definitions to get $\widehat{K_G} = \widehat{G}$.

Equation (3.6) says $K_B = c_B B$ and is analogous to (3.5), yielding $\widehat{K_B} = \widehat{B}$.

Equation (3.7) says $K_U = c_U U$ and is analogous to (3.5)-(3.6), yielding $\widehat{K_U} = \widehat{U}$.

Equation (3.8) says $\bar{K} = K_X + K_Y + K_G + K_B + K_U$. Totally differentiate:

$$0 = dK_X + dK_Y + dK_G + dK_B + dK_U.$$

Then divide by $\bar{K}$, and multiply each term on the right by one (e.g., $\frac{K_X}{K_X}$, $\frac{K_Y}{K_Y}$, $\frac{K_G}{K_G}$, $\frac{K_B}{K_B}$, or $\frac{K_U}{K_U}$) to get:

$$0 = \frac{dK_X}{K_X} \cdot \frac{K_X}{\bar{K}} + \frac{dK_Y}{K_Y} \cdot \frac{K_Y}{\bar{K}} + \frac{dK_G}{K_G} \cdot \frac{K_G}{\bar{K}} + \frac{dK_B}{K_B} \cdot \frac{K_B}{\bar{K}} + \frac{dK_U}{K_U} \cdot \frac{K_U}{\bar{K}}.$$

Use the hat definitions to get: $0 = \widehat{K_X} \theta_{KX} + \widehat{K_Y} \theta_{KY} + \widehat{K_G} \theta_{KG} + \widehat{K_B} \theta_{KB} + \widehat{K_U} \theta_{KU}$, where $\theta_{Ki} \equiv \frac{K_i}{\bar{K}}$ is the fraction of $\bar{K}$ attributable to K_i (for $i = X, Y, G, B, U$).

Equation (3.9) says $p_G G = p_K c_G G + \tau_G G$. Totally differentiate to get $dp_G G + p_G dG = dp_K c_G G + p_K c_G dG + d\tau_G G + \tau_G dG$. Divide by $p_G G$ and simplify to get:

$$\frac{dp_G}{p_G} + \frac{dG}{G} = \frac{dp_K}{p_K} \frac{p_K c_G}{p_G} + \frac{dG}{G} \frac{p_K c_G}{p_G} + \frac{d\tau_G}{\tau_G} \frac{\tau_G}{p_G} + \frac{dG}{G} \frac{\tau_G}{p_G}.$$

Then use the hat definitions to get:

$$\widehat{p_G} + \widehat{G} = \theta_{Gp}(\widehat{p_K} + \widehat{G}) + \theta_{G\tau}(\widehat{\tau_G} + \widehat{G}), \text{ where } \theta_{Gp} \equiv p_K c_G / p_G \text{ and } \theta_{G\tau} \equiv \tau_G / p_G.$$

Equation (3.10) says $p_B B = p_K c_B B + \tau_B B$ and is analogous to (3.9), yielding $\widehat{p_B} + \widehat{B} = \theta_{Bp}(\widehat{p_K} + \widehat{B}) + \theta_{B\tau}(\widehat{\tau_B} + \widehat{B})$, where $\theta_{Bp} \equiv p_K c_B / p_B$ and $\theta_{B\tau} \equiv \tau_B / p_B$.

Equation (3.11a) says $p_U U = p_K c_U U + \tau_U U$ and is analogous to (3.9)-(3.10), yielding $\widehat{p_U} + \widehat{U} = \theta_{Up}(\widehat{p_K} + \widehat{U}) + \theta_{U\tau}(\widehat{\tau_U} + \widehat{U})$. **Equation (3.11b)** sets $\tau_U = \theta_{U\tau} = 0$, so $\widehat{p_U} + \widehat{U} = \widehat{p_K} + \widehat{U}$.

Equation (3.12) says $p_W X = p_G G + p_B B + p_U U$. Totally differentiate, divide by $p_W X$, and multiply each term on the right by one (e.g., $\frac{p_G}{p_G}$, $\frac{G}{G}$, $\frac{p_B}{p_B}$, $\frac{B}{B}$, $\frac{p_U}{p_U}$, or $\frac{U}{U}$) to get:

$$\frac{p_W dX + X dp_W}{p_W X} = \frac{p_G G}{p_W X} \left(\frac{dp_G}{p_G} + \frac{dG}{G} \right) + \frac{p_B B}{p_W X} \left(\frac{dp_B}{p_B} + \frac{dB}{B} \right) + \frac{p_U U}{p_W X} \left(\frac{dp_U}{p_U} + \frac{dU}{U} \right).$$

Use the hat definitions to get:

$$\widehat{X} + \widehat{p_W} = \theta_{XG}(\widehat{p_G} + \widehat{G}) + \theta_{XB}(\widehat{p_B} + \widehat{B}) + \theta_{XU}(\widehat{p_U} + \widehat{U}),$$

where $\theta_{XG} \equiv \frac{p_G G}{p_W X}$, $\theta_{XB} \equiv \frac{p_B B}{p_W X}$, and $\theta_{XU} \equiv \frac{p_U U}{p_W X}$.

Equation (3.13) says that the full cost of spending on X is $p_X^F X = p_X X + p_W X$. Totally differentiate to get $dp_X^F X + p_X^F dX = dp_X X + p_X dX + dp_W X + p_W dX$. Division by $p_X^F X$ and simplification yield:

$$\frac{dp_X^F}{p_X^F} + \frac{dX}{X} = \frac{dp_X}{p_X^F} + \frac{p_X dX}{p_X^F X} + \frac{dp_W}{p_X^F} + \frac{p_W dX}{p_X^F X}.$$

Multiply the first and third terms on the right by one (e.g., $\frac{p_X}{p_X}$ or $\frac{p_W}{p_W}$) and use the hat definitions:

$$\widehat{p_X^F} + \hat{X} = \theta_{p_X}(\widehat{p_X} + \hat{X}) + \theta_{p_W}(\widehat{p_W} + \hat{X}), \text{ where } \theta_{p_X} \equiv \frac{p_X}{p_X^F} \text{ and } \theta_{p_W} \equiv \frac{p_W}{p_X^F}.$$

Equation (3.14) says $X = W(G, B, U)$. Totally differentiate this technology constraint:

$$dX = \frac{\partial W}{\partial G} dG + \frac{\partial W}{\partial B} dB + \frac{\partial W}{\partial U} dU$$

Then divide by X , and multiply each term on the right by one (e.g., $\frac{G}{G}$, $\frac{B}{B}$, or $\frac{U}{U}$) to get:

$$\frac{dX}{X} = \frac{\partial W}{\partial G} \frac{dG}{G} \frac{G}{X} + \frac{\partial W}{\partial B} \frac{dB}{B} \frac{B}{X} + \frac{\partial W}{\partial U} \frac{dU}{U} \frac{U}{X}.$$

Multiply each term on the right by $\frac{p_W}{p_W}$ to get:

$$\frac{dX}{X} = \frac{\partial W}{\partial G} \frac{p_W G}{p_W X} \frac{dG}{G} + \frac{\partial W}{\partial B} \frac{p_W B}{p_W X} \frac{dB}{B} + \frac{\partial W}{\partial U} \frac{p_W U}{p_W X} \frac{dU}{U}$$

In a sense, $W(G, B, U)$ is a production function for disposal with inputs G , B , and U . Thus, $\frac{\partial W}{\partial G}$ is the marginal product of G . Multiplying by p_W gives the value of the marginal product (VMP).

With perfect competition, an input's VMP is its price (e.g., $\frac{\partial W}{\partial G} p_W = p_G$, $\frac{\partial W}{\partial B} p_W = p_B$, and $\frac{\partial W}{\partial U} p_W = p_U$). Use each input price to replace its VMP, and use the hat definitions:

$$\hat{X} = \theta_{XG} \hat{G} + \theta_{XB} \hat{B} + \theta_{XU} \hat{U}, \text{ where } \theta_{XG} \equiv \frac{p_G G}{p_W X}, \theta_{XB} \equiv \frac{p_B B}{p_W X}, \text{ and } \theta_{XU} \equiv \frac{p_U U}{p_W X}.$$

Equations (3.15) and (3.16) follow Allen (1938), Mieszkowski (1972), and Fullerton and Heutel (2007) to model choices among inputs. The firm or household's cost minimization problem is:

$$\min_{G, B, U} p_G G + p_B B + p_U U \quad \text{s.t.} \quad W(G, B, U) = X$$

(where equations (3.9)-(3.11b) imply $p_G = p_K c_G + \tau_G$, $p_B = p_K c_B + \tau_B$, and $p_U = p_K c_U$).

This problem yields demand functions $G = G(p_G, p_B, p_U, X)$, $B = B(p_G, p_B, p_U, X)$, and $U = U(p_G, p_B, p_U, X)$. Those three demand functions are strictly analogous to each other, so we show the derivation only for G . First totally differentiate $G = G(p_G, p_B, p_U, X)$ to get:

$$dG = \frac{\partial G}{\partial p_G} dp_G + \frac{\partial G}{\partial p_B} dp_B + \frac{\partial G}{\partial p_U} dp_U + \frac{\partial G}{\partial X} dX$$

Then divide by G , and multiply each term on the right by one (e.g., $\frac{p_G}{p_G}$, $\frac{p_B}{p_B}$, $\frac{p_U}{p_U}$, or $\frac{X}{X}$) to get:

$$\frac{dG}{G} = \frac{\partial G}{\partial p_G} \frac{p_G}{G} \frac{dp_G}{p_G} + \frac{\partial G}{\partial p_B} \frac{p_B}{G} \frac{dp_B}{p_B} + \frac{\partial G}{\partial p_U} \frac{p_U}{G} \frac{dp_U}{p_U} + \frac{\partial G}{\partial X} \frac{X}{G} \frac{dX}{X}$$

Use hat definitions on G and the prices to get:

$$\hat{G} = \frac{\partial G}{\partial p_G} \frac{p_G}{G} \hat{p}_G + \frac{\partial G}{\partial p_B} \frac{p_B}{G} \hat{p}_B + \frac{\partial G}{\partial p_U} \frac{p_U}{G} \hat{p}_U + \frac{\partial G}{\partial X} \frac{X}{G} \hat{X}$$

In the last term of that equation, CRTS means that the elasticity of demand for each input with respect to output must be one ($\frac{\partial G}{\partial X} \frac{X}{G} = \frac{\partial B}{\partial X} \frac{X}{B} = \frac{\partial U}{\partial X} \frac{X}{U} = 1$). Then we can write the equation as:

$$\hat{G} = a_{GG} \hat{p}_G + a_{GB} \hat{p}_B + a_{GU} \hat{p}_U + \hat{X},$$

where $a_{Gj} \equiv \frac{\partial G}{\partial p_j} \frac{p_j}{G}$ (for $j = G, B, U$). In general, each a_{ij} is the elasticity of demand for input i with respect to a change in the price of input j . Allen (1938) shows that $a_{ij} = e_{ij} \theta_{Xj}$, where e_{ij} is the Allen partial elasticity of substitution, and θ_{Xj} is the share of total disposal expenditure attributable to disposal type j (as just defined above). Allen also shows that $e_{ii} < 0$, $e_{ij} = e_{ji}$, and that $a_{iG} + a_{iB} + a_{iU} = 0$. Thus, at most one of e_{GB} , e_{GU} , and e_{BU} can be negative. Steps to solve for B and U are analogous, so we have:

$$\hat{B} = a_{BG} \hat{p}_G + a_{BB} \hat{p}_B + a_{BU} \hat{p}_U + \hat{X}$$

$$\hat{U} = a_{UG} \hat{p}_G + a_{UB} \hat{p}_B + a_{UU} \hat{p}_U + \hat{X}$$

Since $\hat{X}$ appears in all three equations (for $\hat{G}$, $\hat{B}$, and $\hat{U}$), it can be eliminated by subtracting the third equation (for $\hat{U}$) from each of the first two equations (for $\hat{G}$ and $\hat{B}$):

$$\hat{G} - \hat{U} = (a_{GG} - a_{UG}) \hat{p}_G + (a_{GB} - a_{UB}) \hat{p}_B + (a_{GU} - a_{UU}) \hat{p}_U$$

$$\hat{B} - \hat{U} = (a_{BG} - a_{UG}) \hat{p}_G + (a_{BB} - a_{UB}) \hat{p}_B + (a_{BU} - a_{UU}) \hat{p}_U$$

Substitute Allen elasticities for the a_{ij} to get:

$$\hat{G} - \hat{U} = \theta_{XG}(e_{GG} - e_{UG}) \hat{p}_G + \theta_{XB}(e_{GB} - e_{UB}) \hat{p}_B + \theta_{XU}(e_{GU} - e_{UU}) \hat{p}_U \quad (3.15)$$

$$\hat{B} - \hat{U} = \theta_{XG}(e_{BG} - e_{UG}) \hat{p}_G + \theta_{XB}(e_{BB} - e_{UB}) \hat{p}_B + \theta_{XU}(e_{BU} - e_{UU}) \hat{p}_U, \quad (3.16)$$

where θ_{XG} , θ_{XB} , and θ_{XU} are disposal expenditure shares, and $\theta_{XG} + \theta_{XB} + \theta_{XU} = 1$.

Equation (3.17) says $T = \tau_G G + \tau_B B$, so total differentiation yields $dT = d\tau_G G + dG\tau_G +$

$d\tau_B B + dB\tau_B$. Divide by T to get: $\frac{dT}{T} = \frac{d\tau_G G}{T} + \frac{dG\tau_G}{T} + \frac{d\tau_B B}{T} + \frac{dB\tau_B}{T}$. Multiply each term on the

right by one (e.g., $\frac{\tau_G}{\tau_G}$, $\frac{G}{G}$, $\frac{\tau_B}{\tau_B}$, or $\frac{B}{B}$) to get:

$$\hat{T} = \frac{d\tau_G}{\tau_G} \frac{\tau_G G}{T} + \frac{dG}{G} \frac{G \tau_G}{T} + \frac{d\tau_B}{\tau_B} \frac{\tau_B B}{T} + \frac{dB}{B} \frac{B \tau_B}{T}$$

Use the hat definitions to get:

$$\hat{T} = \theta_{TG}(\widehat{\tau_G} + \hat{G}) + \theta_{TB}(\widehat{\tau_B} + \hat{B}), \text{ where } \theta_{TG} \equiv \frac{\tau_G G}{T} \text{ and } \theta_{TB} \equiv \frac{\tau_B B}{T}.$$

Equation (3.18) says $I = p_K \bar{K} + T$. Total differentiation yields $dI = dp_K \bar{K} + dT$. Then, divide by I to get $\frac{dI}{I} = \frac{dp_K \bar{K}}{I} + \frac{dT}{I}$. Multiply the first term on the right by $\frac{p_K}{p_K}$ and the second term by $\frac{T}{T}$:

$$\frac{dI}{I} = \frac{dp_K}{p_K} \frac{p_K \bar{K}}{I} + \frac{dT}{T} \frac{T}{I}$$

Use the hat definitions to get:

$$\hat{I} = \frac{p_K \bar{K}}{I} \widehat{p_K} + \frac{T}{I} \hat{T}.$$

Equation (3.19) says $\Psi = \Psi(X, Y; nG, nB, nU)$, where the last three entries are not choices of the firm or household. The elasticity of substitution in utility is defined as $\sigma_\Psi > 0$:

$$\sigma_\Psi \equiv \frac{d\left(\frac{X}{Y}\right)}{\left(\frac{X}{Y}\right)} / \frac{d\left(\frac{p_Y}{p_X^F}\right)}{\left(\frac{p_Y}{p_X^F}\right)} = \left(\frac{Y dX - X dY}{Y^2 \left(\frac{X}{Y}\right)} \right) / \left(\frac{p_X^F dp_Y - p_Y dp_X^F}{p_X^{F^2} \left(\frac{p_Y}{p_X^F}\right)} \right) = \left(\frac{dX}{X} - \frac{dY}{Y} \right) / \left(\frac{dp_Y}{p_Y} - \frac{dp_X^F}{p_X^F} \right) = \frac{\hat{X} - \hat{Y}}{\widehat{p_Y} - \widehat{p_X^F}}$$

Rearrange this definition to get a behavioral equation, linear in four of the unknown changes:

$$\hat{X} - \hat{Y} = \sigma_\Psi (\widehat{p_Y} - \widehat{p_X^F})$$

Appendix E – Intermediate Solutions for Changes in Three Types of Waste

We use the 19 linear equations above to solve for unknown changes in quantities and prices. We have ten unknown quantities ($\hat{X}$, $\widehat{K_X}$, $\hat{Y}$, $\widehat{K_Y}$, $\hat{B}$, $\widehat{K_B}$, $\hat{G}$, $\widehat{K_G}$, $\hat{U}$, and $\widehat{K_U}$), eight prices ($\widehat{p_K}$, $\widehat{p_Y}$, $\widehat{p_X}$, $\widehat{p_B}$, $\widehat{p_G}$, $\widehat{p_U}$, $\widehat{p_W}$, and $\widehat{p_X^F}$), two tax rates ($\widehat{\tau_B}$ and $\widehat{\tau_G}$), and two other values ($\hat{T}$ and $\hat{I}$). Of those 22 variables, we set one numeraire ($\widehat{p_K} = 0$), and one exogenous policy shock (e.g., $\widehat{\tau_B} > 0$). We assume the government budget to be a fixed amount ($\hat{T} = 0$), which implies that the exogenous shock $\widehat{\tau_B} > 0$ results in some endogenous change in the other tax rate ($\widehat{\tau_G}$ is unknown). These steps reduce the number of unknowns from 22 to 19, so we can solve for all unknowns using the 19 numbered equations below (3.1 to 3.19 in the text).

- 1) $\hat{X} = \widehat{K_X}$
- 2) $\widehat{p_X} + \hat{X} = \widehat{p_K} + \widehat{K_X}$

- 3) $\hat{Y} = \hat{K}_Y$
- 4) $\hat{p}_Y + \hat{Y} = \hat{p}_K + \hat{K}_Y$
- 5) $\hat{K}_G = \hat{G}$
- 6) $\hat{K}_B = \hat{B}$
- 7) $\hat{K}_U = \hat{U}$
- 8) $0 = \hat{K}_X \theta_{KX} + \hat{K}_Y \theta_{KY} + \hat{K}_G \theta_{KG} + \hat{K}_B \theta_{KB} + \hat{K}_U \theta_{KU}$
- 9) $\hat{p}_G + \hat{G} = \theta_{Gp}(\hat{p}_K + \hat{G}) + \theta_{G\tau}(\hat{\tau}_G + \hat{G})$
- 10) $\hat{p}_B + \hat{B} = \theta_{Bp}(\hat{p}_K + \hat{B}) + \theta_{B\tau}(\hat{\tau}_B + \hat{B})$
- 11) $\hat{p}_U + \hat{U} = \hat{p}_K + \hat{U}$ (using (3.11b))
- 12) $\hat{X} + \hat{p}_W = \theta_{XG}(\hat{p}_G + \hat{G}) + \theta_{XB}(\hat{p}_B + \hat{B}) + \theta_{XU}(\hat{p}_U + \hat{U})$
- 13) $\hat{p}_X^F + \hat{X} = \theta_{pX}(\hat{p}_X + \hat{X}) + \theta_{pW}(\hat{p}_W + \hat{X})$
- 14) $\hat{X} = \theta_{XG}\hat{G} + \theta_{XB}\hat{B} + \theta_{XU}\hat{U}$
- 15) $\hat{G} - \hat{U} = \theta_{XG}(e_{GG} - e_{UG})\hat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\hat{p}_B + \theta_{XU}(e_{GU} - e_{UU})\hat{p}_U$
- 16) $\hat{B} - \hat{U} = \theta_{XG}(e_{BG} - e_{UG})\hat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\hat{p}_B + \theta_{XU}(e_{BU} - e_{UU})\hat{p}_U$
- 17) $\hat{T} = \theta_{TG}(\hat{\tau}_G + \hat{G}) + \theta_{TB}(\hat{\tau}_B + \hat{B})$
- 18) $\hat{I} = \hat{p}_K \frac{p_K \bar{K}}{I} + \frac{T}{I} \hat{T}$
- 19) $\hat{X} - \hat{Y} = \sigma_\psi(\hat{p}_Y - \hat{p}_X^F)$

Equations (3.15) and (3.16) express $\hat{G}$ and $\hat{B}$ as differences from $\hat{U}$, so we solve for $\hat{U}$ first.

Solving for $\hat{U}$:

Step 1E. We define the primary input K as the numeraire, so $\hat{p}_K = 0$. Substitute $\hat{p}_K = 0$ into equations (3.2), (3.4), (3.9), (3.10), (3.11b), and (3.18).

Step 2E. Equation (3.2) minus equation (3.1) yields $\hat{p}_X = 0$.

Step 3E. Equation (3.4) minus equation (3.3) yields $\hat{p}_Y = 0$.

Step 4E. By definition, $\theta_{Gp} + \theta_{G\tau} = 1$. In (3.9), subtract $\hat{G}$ from both sides to get $\hat{p}_G = \theta_{G\tau}\hat{\tau}_G$.

Step 5E. By definition, $\theta_{Bp} + \theta_{B\tau} = 1$. In (3.10), subtract $\hat{B}$ from both sides to get $\hat{p}_B = \theta_{B\tau}\hat{\tau}_B$.

Step 6E. In equation (3.11b), subtract $\hat{U}$ from both sides to get $\hat{p}_U = 0$.

Step 7E. Since $\hat{p}_U = 0$, equation (3.12) minus equation (3.14) yields $\hat{p}_W = \theta_{XB}\hat{p}_B + \theta_{XG}\hat{p}_G$.

Step 8E. By definition, $\theta_{pX} + \theta_{pW} = 1$. In equation (3.13), subtract $\hat{X}$ from both sides and use

$\widehat{p}_X = 0$ from Step 2E to get $\widehat{p}_X^F = \theta_{pW}\widehat{p}_W$.

Step 9E. Substitute $\widehat{X} = \widehat{K}_X$, $\widehat{Y} = \widehat{K}_Y$, $\widehat{K}_G = \widehat{G}$, $\widehat{K}_B = \widehat{B}$, and $\widehat{K}_U = \widehat{U}$ into equation (3.8) to get:

$$0 = \widehat{X}\theta_{KX} + \widehat{Y}\theta_{KY} + \widehat{G}\theta_{KG} + \widehat{B}\theta_{KB} + \widehat{U}\theta_{KU}.$$

Step 10E. Since $\widehat{p}_Y = 0$ from Step 3E, equation (3.19) becomes $\widehat{Y} = \widehat{X} + \sigma_\psi \widehat{p}_X^F$. Substitute $\widehat{Y} = \widehat{X} + \sigma_\psi \widehat{p}_X^F$ into the resulting equation from Step 9E to get:

$$0 = \widehat{X}\theta_{KX} + (\widehat{X} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} + \widehat{G}\theta_{KG} + \widehat{B}\theta_{KB} + \widehat{U}\theta_{KU}.$$

Step 11E. Since $\widehat{p}_U = 0$ from Step 6E, substitute equations (3.15) and (3.16) into the equation resulting from Step 10E to get:

$$\begin{aligned} 0 = & \widehat{X}\theta_{KX} + (\widehat{X} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B + \widehat{U}]\theta_{KG} \\ & + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B + \widehat{U}]\theta_{KB} + \widehat{U}\theta_{KU}. \end{aligned}$$

Step 12E. Substitute equation (3.14) into the resulting equation from Step 11E to get:

$$\begin{aligned} 0 = & (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \theta_{XU}\widehat{U})\theta_{KX} + (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \theta_{XU}\widehat{U} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} \\ & + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B + \widehat{U}]\theta_{KG} \\ & + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B + \widehat{U}]\theta_{KB} + \widehat{U}\theta_{KU}. \end{aligned}$$

Step 13E. Subtract $\theta_{XU}\widehat{U}\theta_{KX}$, $\theta_{XU}\widehat{U}\theta_{KY}$, $\widehat{U}\theta_{KG}$, $\widehat{U}\theta_{KB}$, and $\widehat{U}\theta_{KU}$ from both sides of the resulting equation from Step 12E to get:

$$\begin{aligned} & -\theta_{XU}\widehat{U}\theta_{KX} - \theta_{XU}\widehat{U}\theta_{KY} - \widehat{U}\theta_{KG} - \widehat{U}\theta_{KB} - \widehat{U}\theta_{KU} \\ = & (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B})\theta_{KX} + (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} \\ & + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B]\theta_{KG} \\ & + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B]\theta_{KB}. \end{aligned}$$

Step 14E. By definition, $\theta_{XG} + \theta_{XB} + \theta_{XU} = 1$. Substitute $\theta_{XU} = 1 - \theta_{XG} - \theta_{XB}$ into the resulting equation from Step 13E to get:

$$\begin{aligned} & -(1 - \theta_{XG} - \theta_{XB})\widehat{U}\theta_{KX} - (1 - \theta_{XG} - \theta_{XB})\widehat{U}\theta_{KY} - \widehat{U}\theta_{KG} - \widehat{U}\theta_{KB} - \widehat{U}\theta_{KU} \\ = & (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B})\theta_{KX} + (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} \\ & + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B]\theta_{KG} \\ & + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B]\theta_{KB}. \end{aligned}$$

Step 15E. Expand $(1 - \theta_{XG} - \theta_{XB})$ to get:

$$\begin{aligned} & -\widehat{U}\theta_{KX} - \widehat{U}\theta_{KY} - \widehat{U}\theta_{KG} - \widehat{U}\theta_{KB} - \widehat{U}\theta_{KU} + \theta_{XG}\widehat{U}\theta_{KX} + \theta_{XB}\widehat{U}\theta_{KX} \\ & + \theta_{XG}\widehat{U}\theta_{KY} + \theta_{XB}\widehat{U}\theta_{KY} = (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B})\theta_{KX} + (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \sigma_\psi \widehat{p}_X^F)\theta_{KY} \end{aligned}$$

$$+ [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B]\theta_{KG} \\ + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B]\theta_{KB}.$$

Step 16E. The resource constraint in equation (3.8) implies $\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB} + \theta_{KU} =$

1. The first five terms in the resulting equation from Step 15E sum to $-\widehat{U}$. Collect them to get:

$$-\widehat{U} + \theta_{XG}\widehat{U}\theta_{KX} + \theta_{XB}\widehat{U}\theta_{KB} + \theta_{XG}\widehat{U}\theta_{KY} + \theta_{XB}\widehat{U}\theta_{KY} \\ = (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B})\theta_{KX} + (\theta_{XG}\widehat{G} + \theta_{XB}\widehat{B} + \sigma_\psi\widehat{p}_X^F)\theta_{KY} \\ + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B]\theta_{KG} \\ + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B]\theta_{KB}.$$

Step 17E. Subtract $\theta_{XG}\widehat{U}\theta_{KX}$, $\theta_{XB}\widehat{U}\theta_{KB}$, $\theta_{XG}\widehat{U}\theta_{KY}$, and $\theta_{XB}\widehat{U}\theta_{KY}$ from both sides of the resulting equation from Step 16E to get:

$$-\widehat{U} = [\theta_{XG}(\widehat{G} - \widehat{U}) + \theta_{XB}(\widehat{B} - \widehat{U})]\theta_{KX} + [\theta_{XG}(\widehat{G} - \widehat{U}) + \theta_{XB}(\widehat{B} - \widehat{U}) + \sigma_\psi\widehat{p}_X^F]\theta_{KY} \\ + [\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B]\theta_{KG} \\ + [\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B]\theta_{KB}.$$

Step 18E. In the resulting equation from Step 17E, replace $\widehat{G} - \widehat{U}$ and $\widehat{B} - \widehat{U}$ with equations (3.15) and (3.16), and then factor out each Allen elasticity difference to get:

$$-\widehat{U} = \theta_{XG}\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G\theta_{KX} + \theta_{XG}\theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G\theta_{KY} + \theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G\theta_{KG} \\ + \theta_{XB}\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G\theta_{KX} + \theta_{XB}\theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G\theta_{KY} + \theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G\theta_{KB} \\ + \theta_{XG}\theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B\theta_{KX} + \theta_{XG}\theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B\theta_{KY} + \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B\theta_{KB} \\ + \theta_{XB}\theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B\theta_{KX} + \theta_{XB}\theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B\theta_{KY} + \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B\theta_{KB} \\ + \sigma_\psi\widehat{p}_X^F\theta_{KY}.$$

Step 19E. Substitute $\widehat{p}_X^F = \theta_{pW}\widehat{p}_W$ from Step 8E, substitute $\widehat{p}_W = \theta_{XB}\widehat{p}_B + \theta_{XG}\widehat{p}_G$ from Step 7E, and then combine like terms in $\widehat{p}_B$ and $\widehat{p}_G$ to get:

$$-\widehat{U} = [(e_{GG} - e_{UG})(\theta_{XG}\theta_{KX} + \theta_{XG}\theta_{KY} + \theta_{KG}) + (e_{BG} - e_{UG})(\theta_{XB}\theta_{KX} + \theta_{XB}\theta_{KY} + \theta_{KB}) \\ + \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\widehat{p}_G \\ + [(e_{GB} - e_{UB})(\theta_{XG}\theta_{KX} + \theta_{XG}\theta_{KY} + \theta_{KG}) + (e_{BB} - e_{UB})(\theta_{XB}\theta_{KX} + \theta_{XB}\theta_{KY} + \theta_{KB}) \\ + \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\widehat{p}_B.$$

Step 20E. Substitute $\widehat{p}_G = \theta_{G\tau}\widehat{\tau}_G$ from Step 4E and substitute $\widehat{p}_B = \theta_{B\tau}\widehat{\tau}_B$ from Step 5E, and then multiply both sides by -1 to get:

$$\begin{aligned}\widehat{U} = & [(e_{UG} - e_{GG})(\theta_{XG}\theta_{KX} + \theta_{XG}\theta_{KY} + \theta_{KG}) + (e_{UG} - e_{BG})(\theta_{XB}\theta_{KX} + \theta_{XB}\theta_{KY} + \theta_{KB}) \\ & - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ & + [(e_{UB} - e_{GB})(\theta_{XG}\theta_{KX} + \theta_{XG}\theta_{KY} + \theta_{KG}) + (e_{UB} - e_{BB})(\theta_{XB}\theta_{KX} + \theta_{XB}\theta_{KY} + \theta_{KB}) \\ & - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 21E. Expand the resulting equation from Step 20E so that each Allen elasticity separately multiplies a group of factor shares to get:

$$\begin{aligned}\widehat{U} = & [e_{UG}((\theta_{XG} + \theta_{XB})(\theta_{KX} + \theta_{KY}) + \theta_{KG} + \theta_{KB}) - e_{GG}(\theta_{XG}(\theta_{KX} + \theta_{KY}) + \theta_{KG}) \\ & - e_{BG}(\theta_{XB}(\theta_{KX} + \theta_{KY}) + \theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ & + [e_{UB}((\theta_{XG} + \theta_{XB})(\theta_{KX} + \theta_{KY}) + \theta_{KG} + \theta_{KB}) - e_{GB}(\theta_{XG}(\theta_{KX} + \theta_{KY}) + \theta_{KG}) \\ & - e_{BB}(\theta_{XB}(\theta_{KX} + \theta_{KY}) + \theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 22E. By definition, $\theta_{XG} + \theta_{XB} + \theta_{XU} = 1$. Substitute $1 - \theta_{XU}$ for $\theta_{XG} + \theta_{XB}$ to get:

$$\begin{aligned}\widehat{U} = & [e_{UG}((1 - \theta_{XU})(\theta_{KX} + \theta_{KY}) + \theta_{KG} + \theta_{KB}) - e_{GG}(\theta_{XG}(\theta_{KX} + \theta_{KY}) + \theta_{KG}) \\ & - e_{BG}(\theta_{XB}(\theta_{KX} + \theta_{KY}) + \theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ & + [e_{UB}((1 - \theta_{XU})(\theta_{KX} + \theta_{KY}) + \theta_{KG} + \theta_{KB}) - e_{GB}(\theta_{XG}(\theta_{KX} + \theta_{KY}) + \theta_{KG}) \\ & - e_{BB}(\theta_{XB}(\theta_{KX} + \theta_{KY}) + \theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 23E. Expand each parenthesis except $(\theta_{KX} + \theta_{KY})$, and then reorganize terms to get:

$$\begin{aligned}\widehat{U} = & [e_{UG}(\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB}) - (\theta_{KX} + \theta_{KY})(e_{UG}\theta_{XU} + e_{GG}\theta_{XG} + e_{BG}\theta_{XB}) - e_{GG}\theta_{KG} \\ & - e_{BG}\theta_{KB} - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ & + [e_{UB}(\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB}) - (\theta_{KX} + \theta_{KY})(e_{UB}\theta_{XU} + e_{GB}\theta_{XG} + e_{BB}\theta_{XB}) - e_{GB}\theta_{KG} \\ & - e_{BB}\theta_{KB} - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 24E. The resource constraint implies $\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB} + \theta_{KU} = 1$. Substitute $1 - \theta_{KU}$ for $\theta_{KX} + \theta_{KY} + \theta_{KG} + \theta_{KB}$ to get:

$$\begin{aligned}\widehat{U} = & [e_{UG}(1 - \theta_{KU}) - (\theta_{KX} + \theta_{KY})(e_{UG}\theta_{XU} + e_{GG}\theta_{XG} + e_{BG}\theta_{XB}) - e_{GG}\theta_{KG} - e_{BG}\theta_{KB} \\ & - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ & + [e_{UB}(1 - \theta_{KU}) - (\theta_{KX} + \theta_{KY})(e_{UB}\theta_{XU} + e_{GB}\theta_{XG} + e_{BB}\theta_{XB}) - e_{GB}\theta_{KG} - e_{BB}\theta_{KB} \\ & - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 25E. The adding-up constraint for the Allen elasticities says that $a_{iG} + a_{iB} + a_{iU} = 0$, where $a_{ij} = e_{ij}\theta_{Xj}$. They are also symmetric, so $e_{ij} = e_{ji}$. Substitute zero for both $e_{UG}\theta_{XU} +$

$e_{GG}\theta_{XG} + e_{BG}\theta_{XB}$ and $e_{UB}\theta_{XU} + e_{GB}\theta_{XG} + e_{BB}\theta_{XB}$ to get:

$$\begin{aligned}\widehat{U} &= [e_{UG}(1 - \theta_{KU}) - e_{GG}\theta_{KG} - e_{BG}\theta_{KB} - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ &\quad + [e_{UB}(1 - \theta_{KU}) - e_{GB}\theta_{KG} - e_{BB}\theta_{KB} - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B.\end{aligned}$$

Step 26E. Define ε_{UB} as the intermediate elasticity for the percent change in U from a one percent change only in τ_B . Analogously, define ε_{UG} as the intermediate elasticity for the percent change in U from a one percent change only in τ_G .

$$\begin{aligned}\varepsilon_{UB} &\equiv [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau} \\ \varepsilon_{UG} &\equiv [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\end{aligned}$$

Step 27E. Substitute ε_{UB} and ε_{UG} into the resulting equation for $\widehat{U}$ from Step 25E to get:

$$\widehat{U} = \varepsilon_{UB}\widehat{\tau}_B + \varepsilon_{UG}\widehat{\tau}_G \quad (4.1)$$

Solving for $\widehat{B}$:

Step 28E. Since $\widehat{p}_U = 0$ from Step 6E, equation (3.16) becomes:

$$\widehat{B} - \widehat{U} = \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B + \theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G.$$

Step 29E. Add $\widehat{U}$ to both sides and substitute the resulting equation from Step 25E to get:

$$\begin{aligned}\widehat{B} &= \theta_{XB}(e_{BB} - e_{UB})\widehat{p}_B + \theta_{XG}(e_{BG} - e_{UG})\widehat{p}_G \\ &\quad + [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &\quad + [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G.\end{aligned}$$

Step 30E. Substitute $\widehat{p}_B = \theta_{B\tau}\widehat{\tau}_B$ from Step 5E and substitute $\widehat{p}_G = \theta_{G\tau}\widehat{\tau}_G$ from Step 4E to get:

$$\begin{aligned}\widehat{B} &= (e_{BB} - e_{UB})\theta_{XB}\theta_{B\tau}\widehat{\tau}_B + (e_{BG} - e_{UG})\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ &\quad + [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &\quad + [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G.\end{aligned}$$

Step 31E. Combine like terms to get:

$$\begin{aligned}\widehat{B} &= [e_{UB}(-\theta_{KU}) + e_{BB}(1 - \theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &\quad + [e_{UG}(-\theta_{KU}) - e_{GG}(\theta_{KG}) + e_{BG}(1 - \theta_{KB}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G.\end{aligned}$$

Step 32E. Define ε_{BB} as the intermediate elasticity for the percent change in B from a one percent change only in τ_B . Analogously, define ε_{BG} as the intermediate elasticity for the percent change in B from a one percent change only in τ_G .

$$\varepsilon_{BB} \equiv [e_{UB}(-\theta_{KU}) + e_{BB}(1 - \theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_\psi\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}$$

$$\varepsilon_{BG} \equiv [e_{UG}(-\theta_{KU}) - e_{GG}(\theta_{KG}) + e_{BG}(1 - \theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}$$

Step 33E. Substitute ε_{BB} and ε_{BG} into $\widehat{B}$ from Step 31E to get:

$$\widehat{B} = \varepsilon_{BB}\widehat{\tau}_B + \varepsilon_{BG}\widehat{\tau}_G \quad (4.2)$$

Solving for $\widehat{G}$:

Step 34E. Since $\widehat{p}_U = 0$ from Step 6E, equation (3.15) becomes:

$$\widehat{G} - \widehat{U} = \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B + \theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G.$$

Step 35E. Add $\widehat{U}$ to both sides and substitute the resulting equation from Step 25E to get:

$$\begin{aligned} \widehat{G} &= \theta_{XB}(e_{GB} - e_{UB})\widehat{p}_B + \theta_{XG}(e_{GG} - e_{UG})\widehat{p}_G \\ &+ [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &+ [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G. \end{aligned}$$

Step 36E. Substitute $\widehat{p}_B = \theta_{B\tau}\widehat{\tau}_B$ from Step 5E and substitute $\widehat{p}_G = \theta_{G\tau}\widehat{\tau}_G$ from Step 4E to get:

$$\begin{aligned} \widehat{G} &= (e_{GB} - e_{UB})\theta_{XB}\theta_{B\tau}\widehat{\tau}_B + (e_{GG} - e_{UG})\theta_{XG}\theta_{G\tau}\widehat{\tau}_G \\ &+ [e_{UB}(1 - \theta_{KU}) - e_{BB}(\theta_{KB}) - e_{GB}(\theta_{KG}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &+ [e_{UG}(1 - \theta_{KU}) - e_{GG}(\theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G. \end{aligned}$$

Step 37E. Combine like terms to get:

$$\begin{aligned} \widehat{G} &= [e_{UB}(-\theta_{KU}) - e_{BB}(\theta_{KB}) + e_{GB}(1 - \theta_{KG}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau}\widehat{\tau}_B \\ &+ [e_{UG}(-\theta_{KU}) + e_{GG}(1 - \theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau}\widehat{\tau}_G. \end{aligned}$$

Step 38E. Define ε_{GB} as the intermediate elasticity for the percent change in G from a one percent change only in τ_B . Analogously, define ε_{GG} as the intermediate elasticity for the percent change in G from a one percent change only in τ_G .

$$\begin{aligned} \varepsilon_{GB} &\equiv [e_{UB}(-\theta_{KU}) - e_{BB}(\theta_{KB}) + e_{GB}(1 - \theta_{KG}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XB}\theta_{B\tau} \\ \varepsilon_{GG} &\equiv [e_{UG}(-\theta_{KU}) + e_{GG}(1 - \theta_{KG}) - e_{BG}(\theta_{KB}) - \sigma_{\psi}\theta_{pW}\theta_{KY}]\theta_{XG}\theta_{G\tau} \end{aligned}$$

Step 39E. Substitute ε_{GB} and ε_{GG} into $\widehat{G}$ from Step 37E to get:

$$\widehat{G} = \varepsilon_{GB}\widehat{\tau}_B + \varepsilon_{GG}\widehat{\tau}_G \quad (4.3)$$

Appendix F – Closed-form Solution for $\widehat{\tau}_G$ in Terms of Parameters and Exogenous $\widehat{\tau}_B$

Using equations (3.17), (4.2), and (4.3), we solve for $\widehat{\tau}_G$ in terms of the exogenous shock $\widehat{\tau}_B$ to get closed-form solutions for $\widehat{G}$, $\widehat{B}$, and $\widehat{U}$. We set $\widehat{\tau}_B > 0$ as the exogenous shock and $\widehat{T} = 0$ with a fixed government budget, so we are left with three unknowns ($\widehat{G}$, $\widehat{\tau}_G$, and $\widehat{B}$). Use the

following three equations to solve for three unknowns:

$$\hat{T} = \theta_{TG}(\widehat{\tau}_G + \hat{G}) + \theta_{TB}(\widehat{\tau}_B + \hat{B}) \quad (3.17)$$

$$\hat{B} = \varepsilon_{BB}\widehat{\tau}_B + \varepsilon_{BG}\widehat{\tau}_G \quad (4.2)$$

$$\hat{G} = \varepsilon_{GB}\widehat{\tau}_B + \varepsilon_{GG}\widehat{\tau}_G \quad (4.3)$$

Step 1F. Substitute equations (4.2) and (4.3) for $\hat{B}$ and $\hat{G}$ into equation (3.17) for $\hat{T}$:

$$\hat{T} = \theta_{TG}\widehat{\tau}_G + \theta_{TG}(\varepsilon_{GB}\widehat{\tau}_B + \varepsilon_{GG}\widehat{\tau}_G) + \theta_{TB}\widehat{\tau}_B + \theta_{TB}(\varepsilon_{BB}\widehat{\tau}_B + \varepsilon_{BG}\widehat{\tau}_G).$$

Step 2F. Assume a fixed revenue requirement, so set $\hat{T} = 0$:

$$0 = \theta_{TG}\widehat{\tau}_G + \theta_{TG}(\varepsilon_{GB}\widehat{\tau}_B + \varepsilon_{GG}\widehat{\tau}_G) + \theta_{TB}\widehat{\tau}_B + \theta_{TB}(\varepsilon_{BB}\widehat{\tau}_B + \varepsilon_{BG}\widehat{\tau}_G).$$

Step 3F. Subtract $\theta_{TG}\widehat{\tau}_G$, $\theta_{TG}\varepsilon_{GG}\widehat{\tau}_G$, and $\theta_{TB}\varepsilon_{BG}\widehat{\tau}_G$ from both sides to get:

$$-\theta_{TG}\widehat{\tau}_G - \theta_{TG}\varepsilon_{GG}\widehat{\tau}_G - \theta_{TB}\varepsilon_{BG}\widehat{\tau}_G = \theta_{TG}\varepsilon_{GB}\widehat{\tau}_B + \theta_{TB}\widehat{\tau}_B + \theta_{TB}\varepsilon_{BB}\widehat{\tau}_B.$$

Step 4F. Factor out $\widehat{\tau}_G$ on the left-hand side and $\widehat{\tau}_B$ on the right-hand side to get:

$$-(\theta_{TG} + \theta_{TG}\varepsilon_{GG} + \theta_{TB}\varepsilon_{BG})\widehat{\tau}_G = (\theta_{TG}\varepsilon_{GB} + \theta_{TB} + \theta_{TB}\varepsilon_{BB})\widehat{\tau}_B.$$

Step 5F. Define E_{TG} as the GE elasticity of total tax revenue T for a change in the tax rate on good disposal τ_G . Analogously, E_{TB} is the GE elasticity for the percent change in T from a one percent change in τ_B . We assume that both tax rates are on the normal side of the Laffer curve, so that $\widehat{\tau}_B > 0$ raises revenue that can be used to cut the other rate ($\widehat{\tau}_G < 0$).

$$E_{TG} \equiv \theta_{TG} + \theta_{TG}\varepsilon_{GG} + \theta_{TB}\varepsilon_{BG}$$

$$E_{TB} \equiv \theta_{TG}\varepsilon_{GB} + \theta_{TB} + \theta_{TB}\varepsilon_{BB}$$

Step 6F. Substitute E_{TG} and E_{TB} into the resulting equation from Step 4F to get:

$$E_{TG}\widehat{\tau}_G = -E_{TB}\widehat{\tau}_B \quad (4.4)$$

Step 7F. Divide both sides by $-E_{TG}$ to get:

$$\widehat{\tau}_G = -\frac{E_{TB}}{E_{TG}}\widehat{\tau}_B.$$

Appendix G – Closed-form Solutions for $\hat{G}$, $\hat{B}$, and $\hat{U}$ with Exogenous Shock $\widehat{\tau}_B$

Step 1G. Substitute the resulting equation from Step 7F into equations (4.1), (4.2), and (4.3) to get:

$$\hat{G} = \varepsilon_{GB}\widehat{\tau}_B - \varepsilon_{GG}\frac{E_{TB}}{E_{TG}}\widehat{\tau}_B, \quad \hat{B} = \varepsilon_{BB}\widehat{\tau}_B - \varepsilon_{BG}\frac{E_{TB}}{E_{TG}}\widehat{\tau}_B, \quad \text{and} \quad \hat{U} = \varepsilon_{UB}\widehat{\tau}_B - \varepsilon_{UG}\frac{E_{TB}}{E_{TG}}\widehat{\tau}_B.$$

Step 2G. Factor out $\widehat{\tau}_B$ in each equation to get:

$$\hat{G} = \left(\varepsilon_{GB} - \varepsilon_{GG}\frac{E_{TB}}{E_{TG}}\right)\widehat{\tau}_B, \quad \hat{B} = \left(\varepsilon_{BB} - \varepsilon_{BG}\frac{E_{TB}}{E_{TG}}\right)\widehat{\tau}_B, \quad \text{and} \quad \hat{U} = \left(\varepsilon_{UB} - \varepsilon_{UG}\frac{E_{TB}}{E_{TG}}\right)\widehat{\tau}_B.$$

Step 3G. Define E_{iB} as the full GE elasticity for the percentage change in disposal i from a one percent change in the tax on the bad disposal, τ_B . Using these three definitions, we have:

$$\hat{G} = E_{GB}\hat{\tau}_B \text{ where } E_{GB} \equiv \varepsilon_{GB} - \varepsilon_{GG} \frac{E_{TB}}{E_{TG}} \quad (4.5)$$

$$\hat{B} = E_{BB}\hat{\tau}_B \text{ where } E_{BB} \equiv \varepsilon_{BB} - \varepsilon_{BG} \frac{E_{TB}}{E_{TG}} \quad (4.6)$$

$$\hat{U} = E_{UB}\hat{\tau}_B \text{ where } E_{UB} \equiv \varepsilon_{UB} - \varepsilon_{UG} \frac{E_{TB}}{E_{TG}} \quad (4.7)$$

Appendix H – Welfare Analysis when All Three Disposal Methods Are Taxable

Use equation (3.11a), which includes a tax on the ugly (τ_U), instead of (3.11b) to derive the FBO Pigovian solution when all three polluting inputs have a policy variable. We previously defined household utility in Equation (3.19) as:

$$\Psi = \Psi(X, Y; nG, nB, nU).$$

First solve the household's problem (and later the social planner's problem):

Step 1H. Substitute the waste function $X = W(G, B, U)$ from equation (3.14) into the household utility function. Do not alter the budget constraint yet. The household chooses G, B, U , and Y to maximize the Lagrangean:

$$\mathcal{L} = \Psi[W(G, B, U), Y; nG, nB, nU] + \lambda[I - p_X^F X - p_Y Y] \quad (5.0)$$

Step 2H. Substitute $p_X X + p_W X$ for $p_X^F X$ from equation (3.13) to get:

$$\mathcal{L} = \Psi(W(G, B, U), Y; nG, nB, nU) + \lambda[I - p_X X - p_W X - p_Y Y].$$

Step 3H. Substitute $p_G G + p_B B + p_U U$ for $p_W X$ from equation (3.12) to get:

$$\mathcal{L} = \Psi(W(G, B, U), Y; nG, nB, nU) + \lambda[I - p_X X - p_G G - p_B B - p_U U - p_Y Y].$$

Step 4H. Substitute $W(G, B, U)$ for X in the budget constraint to get:

$$\begin{aligned} \mathcal{L} = & \Psi(W(G, B, U), Y; nG, nB, nU) \\ & + \lambda[I - p_X W(G, B, U) - p_G G - p_B B - p_U U - p_Y Y]. \end{aligned}$$

Step 5H. Use $p_K c_G G + \tau_G G$ for $p_G G$, use $p_K c_B B + \tau_B B$ for $p_B B$, and use $p_K c_U U + \tau_U U$ for $p_U U$ from equations (3.9), (3.10), and (3.11a) to get:

$$\begin{aligned} \mathcal{L} = & \Psi(W(G, B, U), Y; nG, nB, nU) + \lambda[I - p_X W(G, B, U) \\ & - p_K c_G G - \tau_G G - p_K c_B B - \tau_B B - p_K c_U U - \tau_U U - p_Y Y]. \end{aligned}$$

Step 6H. Differentiate the Lagrangean with respect to G , taking nG, nB , and nU as given. Then set the derivative equal to zero, and solve the first-order condition for λ to get:

$$\frac{\partial \Psi \partial W}{\partial X \partial G} - \lambda \left[p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G \right] = 0$$

$$\frac{\partial \Psi \partial W}{\partial X \partial G} = \lambda \left[p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G \right]$$

$$\frac{\frac{\partial \Psi \partial W}{\partial X \partial G}}{p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G} = \lambda$$

Step 7H. Repeat for B , U , and Y to get:

$$\frac{\partial \Psi \partial W}{\partial X \partial B} - \lambda \left[p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B \right] = 0$$

$$\frac{\partial \Psi \partial W}{\partial X \partial B} = \lambda \left[p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B \right]$$

$$\frac{\frac{\partial \Psi \partial W}{\partial X \partial B}}{p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B} = \lambda$$

$$\frac{\partial \Psi \partial W}{\partial X \partial U} - \lambda \left[p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U \right] = 0$$

$$\frac{\partial \Psi \partial W}{\partial X \partial U} = \lambda \left[p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U \right]$$

$$\frac{\frac{\partial \Psi \partial W}{\partial X \partial U}}{p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U} = \lambda$$

$$\frac{\partial \Psi}{\partial Y} - \lambda [p_Y] = 0$$

$$\frac{\partial \Psi}{\partial Y} = \lambda [p_Y]$$

$$\frac{\frac{\partial \Psi}{\partial Y}}{p_Y} = \lambda$$

Step 8H. After solving first-order conditions, the household sets the marginal utility of income:

$$\lambda = \frac{\frac{\partial \Psi \partial W}{\partial X \partial G}}{(p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G)} = \frac{\frac{\partial \Psi \partial W}{\partial X \partial B}}{(p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B)} = \frac{\frac{\partial \Psi \partial W}{\partial X \partial U}}{(p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U)} = \frac{\frac{\partial \Psi}{\partial Y}}{p_Y}.$$

The social planner's problem:

Step 9H. Totally differentiate utility in equation (3.19) to get:

$$\begin{aligned} d\Psi &= \frac{\partial \Psi}{\partial X} \frac{\partial W}{\partial G} dG + \frac{\partial \Psi}{\partial X} \frac{\partial W}{\partial B} dB + \frac{\partial \Psi}{\partial X} \frac{\partial W}{\partial U} dU \\ &+ \frac{\partial \Psi}{\partial Y} dY + \frac{\partial \Psi}{\partial nG} ndG + \frac{\partial \Psi}{\partial nB} ndB + \frac{\partial \Psi}{\partial nU} ndU. \end{aligned}$$

Step 10H. The omniscient social planner then substitutes the household's first-order conditions from Steps 6H and 7H to get:

$$d\Psi = \lambda \left(p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G \right) dG + \lambda \left(p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B \right) dB \\ + \lambda \left(p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U \right) dU + \lambda p_Y dY + \frac{\partial \Psi}{\partial nG} ndG + \frac{\partial \Psi}{\partial nB} ndB + \frac{\partial \Psi}{\partial nU} ndU.$$

Step 11H. Totally differentiate the economy's resource constraint in equation (3.8), $\bar{K} = K_X + K_Y + K_G + K_B + K_U$, to get $d\bar{K} = dK_X + dK_Y + dK_G + dK_B + dK_U$. The left-hand side reduces to zero because $\bar{K}$ is fixed, so $0 = dK_X + dK_Y + dK_G + dK_B + dK_U$.

Step 12H. Multiply each term by the price of the primary input p_K to get:

$$0 = p_K dK_X + p_K dK_Y + p_K dK_G + p_K dK_B + p_K dK_U.$$

Step 13H. Substitute the zero-profit conditions from equations (3.2) and (3.4) to get:

$$0 = p_X dX + p_Y dY + p_K dK_G + p_K dK_B + p_K dK_U.$$

Step 14H. Total differentiation of equation (3.5) yields $dK_G = c_G dG$. Differentiations of equations (3.6) and (3.7) are analogous. Substitute the differentials to get:

$$0 = p_X dX + p_Y dY + p_K c_G dG + p_K c_B dB + p_K c_U dU.$$

Step 15H. Subtract $p_Y dY$ from both sides, and then multiply both sides by -1 to get:

$$p_Y dY = -p_X dX - p_K c_G dG - p_K c_B dB - p_K c_U dU.$$

Step 16H. Substitute $dX = \frac{\partial W}{\partial G} dG + \frac{\partial W}{\partial B} dB + \frac{\partial W}{\partial U} dU$ from equation (3.14) to get:

$$p_Y dY = -p_X \frac{\partial W}{\partial G} dG - p_X \frac{\partial W}{\partial B} dB - p_X \frac{\partial W}{\partial U} dU - p_K c_G dG - p_K c_B dB - p_K c_U dU.$$

Step 17H. Substitute the resulting equation from Step 16H for $p_Y dY$ into the resulting equation for $d\Psi$ from Step 10H to get:

$$d\Psi = \lambda \left(p_X \frac{\partial W}{\partial G} + p_K c_G + \tau_G \right) dG + \lambda \left(p_X \frac{\partial W}{\partial B} + p_K c_B + \tau_B \right) dB \\ + \lambda \left(p_X \frac{\partial W}{\partial U} + p_K c_U + \tau_U \right) dU - \lambda p_X \frac{\partial W}{\partial G} dG - \lambda p_X \frac{\partial W}{\partial B} dB - \lambda p_X \frac{\partial W}{\partial U} dU - \lambda p_K c_G dG \\ - \lambda p_K c_B dB - \lambda p_K c_U dU + \frac{\partial \Psi}{\partial nG} ndG + \frac{\partial \Psi}{\partial nB} ndB + \frac{\partial \Psi}{\partial nU} ndU.$$

Step 18H. Cancel the negative terms against the terms inside the parentheses to get:

$$d\Psi = \lambda(\tau_G) dG + \lambda(\tau_B) dB + \lambda(\tau_U) dU + \frac{\partial \Psi}{\partial nG} ndG + \frac{\partial \Psi}{\partial nB} ndB + \frac{\partial \Psi}{\partial nU} ndU.$$

Step 19H. Divide by λ , the marginal utility of income, to convert $d\Psi$ into the dollar value of the change in utility ($\frac{d\Psi}{\lambda}$):

$$\frac{d\Psi}{\lambda} = (\tau_G)dG + (\tau_B)dB + (\tau_U)dU + \frac{n}{\lambda} \frac{\partial \Psi}{\partial nG} dG + \frac{n}{\lambda} \frac{\partial \Psi}{\partial nB} dB + \frac{n}{\lambda} \frac{\partial \Psi}{\partial nU} dU.$$

Step 20H. Define the marginal external damages (MED) per unit of good, bad, and ugly disposal:

$$\mu_G \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nG}, \mu_B \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nB}, \text{ and } \mu_U \equiv -\frac{n}{\lambda} \frac{\partial \Psi}{\partial nU}.$$

Step 21H. Substitute μ_G , μ_B , and μ_U into the resulting equation from Step 19H to get:

$$\frac{d\Psi}{\lambda} = (\tau_G)dG + (\tau_B)dB + (\tau_U)dU - \mu_G dG - \mu_B dB - \mu_U dU.$$

Step 22H. Combine like terms to get:

$$\frac{d\Psi}{\lambda} = (\tau_G - \mu_G)dG + (\tau_B - \mu_B)dB + (\tau_U - \mu_U)dU.$$

Step 23H. Multiply $(\tau_G - \mu_G)dG$ by $\frac{G}{G}$, $(\tau_B - \mu_B)dB$ by $\frac{B}{B}$, and $(\tau_U - \mu_U)dU$ by $\frac{U}{U}$, and use the hat definitions to get:

$$\frac{d\Psi}{\lambda} = (\tau_G - \mu_G)G\hat{G} + (\tau_B - \mu_B)B\hat{B} + (\tau_U - \mu_U)U\hat{U}.$$

Step 24H. Divide both sides by income I , to express that cash-equivalent change as a fraction of income. Then our measure of the relative change in welfare is:

$$\frac{d\Psi}{\lambda I} = (\tau_G - \mu_G)\frac{G}{I}\hat{G} + (\tau_B - \mu_B)\frac{B}{I}\hat{B} + (\tau_U - \mu_U)\frac{U}{I}\hat{U} \quad (1.2)$$

Utility is maximized when $d\Psi = 0$. We assume that the second-order sufficient conditions hold.

The FBO Pigovian solution occurs when each policy variable (τ_G , τ_B , and τ_U) is set equal to its respective marginal external damage (μ_G , μ_B , and μ_U).

Appendix I – Welfare Analysis with No Tax on Ugly Disposal

When ugly disposal cannot be taxed ($\tau_U = 0$), equation (1.2) becomes:

$$\frac{d\Psi}{\lambda I} = (\tau_G - \mu_G)\frac{G}{I}\hat{G} + (\tau_B - \mu_B)\frac{B}{I}\hat{B} - \mu_U\frac{U}{I}\hat{U} \quad (1.2')$$

Find the “best constrained” tax rate τ_B^* given a fixed non-optimal τ_G :

Continuing with the case where $\widehat{\tau_B}$ is the exogenous shock, we first solve for the best constrained tax rate τ_B^* with two policy constraints – where τ_U is fixed at zero *and* τ_G is fixed at some arbitrary (potentially non-optimal) level. Set the change in utility in (1.2') to zero to solve for the planner's τ_B that maximizes welfare. That is, reorganize terms so that the pre-existing τ_B in (1.2') is by itself on the left-hand side.

Step 1I. Subtract $(\tau_B - \mu_B) \frac{B}{I} \hat{B}$ from both sides of (1.2'):

$$-(\tau_B - \mu_B) \frac{B}{I} \hat{B} = (\tau_G - \mu_G) \frac{G}{I} \hat{G} - \mu_U \frac{U}{I} \hat{U}$$

Step 2I. Divide both sides by $\hat{B}$ and multiply both sides by I :

$$-(\tau_B - \mu_B) B = (\tau_G - \mu_G) G \frac{\hat{G}}{\hat{B}} - \mu_U U \frac{\hat{U}}{\hat{B}}$$

Step 3I. Divide both sides by B (and then remove the parentheses on the left-hand side):

$$-\tau_B + \mu_B = (\tau_G - \mu_G) \frac{G}{B} \frac{\hat{G}}{\hat{B}} - \mu_U \frac{U}{B} \frac{\hat{U}}{\hat{B}}$$

Step 4I. Subtract μ_B from both sides, and notice that all μ_i on the right side have the same sign:

$$-\tau_B = -\mu_B + (\tau_G - \mu_G) \frac{G}{B} \frac{\hat{G}}{\hat{B}} - \mu_U \frac{U}{B} \frac{\hat{U}}{\hat{B}}$$

Step 5I. Multiply both sides by -1 to get the best constrained tax rate τ_B^* when $\tau_U = 0$, and the other tax rate is fixed:

$$\tau_B^* = \mu_B + (\mu_G - \tau_G) \frac{G}{B} \frac{\hat{G}}{\hat{B}} + \mu_U \frac{U}{B} \frac{\hat{U}}{\hat{B}}$$

Step 6I. Substitute equations (4.5), (4.6), and (4.7) for $\hat{G}$, $\hat{B}$, and $\hat{U}$, and cancel the exogenous $\widehat{\tau_B}$ from the numerator and denominator of each ratio to arrive at a closed-form solution for τ_B^* :

$$\tau_B^* = \mu_B + \underbrace{(\mu_G - \tau_G) \frac{G}{B} \left(\frac{E_{GB}}{E_{BB}} \right)}_{\text{Revenue Term}} + \underbrace{\mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)}_{\text{Corlett-Hague Term}}$$

This best constrained tax τ_B^* is comprised of three MED parameters (μ_G , μ_B , and μ_U), two ratios of pre-existing polluting inputs ($\frac{G}{B}$ and $\frac{U}{B}$), the fixed existing τ_G , and three GE elasticity parameters (E_{GB} , E_{BB} , and E_{UB}). Each GE elasticity combines revenue elasticities (E_{TB} and E_{TG}) and intermediate elasticities (ε_{GG} , ε_{GB} , ε_{BG} , ε_{BB} , ε_{UG} , and ε_{UB}), and each intermediate elasticity combines PE elasticities (all of the e_{ij}).

Similar steps to solve for the best constrained tax rate τ_G^* given a fixed non-optimal τ_B :

Here, we also start with equation (1.2') above, but instead of assuming an exogenous shock $\widehat{\tau_B}$, we consider an exogenous shock $\widehat{\tau_G}$.

Step 7I. Subtract $(\tau_G - \mu_G) \frac{G}{I} \hat{G}$ from both sides of (1.2'):

$$-(\tau_G - \mu_G) \frac{G}{I} \hat{G} = (\tau_B - \mu_B) \frac{B}{I} \hat{B} - \mu_U \frac{U}{I} \hat{U}$$

Step 8I. Multiply both sides by $I/(G\hat{G})$, and remove parentheses on the left side:

$$-\tau_G + \mu_G = (\tau_B - \mu_B) \frac{B}{G} \frac{\hat{B}}{\hat{G}} - \mu_U \frac{U}{G} \frac{\hat{U}}{\hat{G}}$$

Step 9I. Subtract μ_G from both sides:

$$-\tau_G = -\mu_G + (\tau_B - \mu_B) \frac{B}{G} \frac{\hat{B}}{\hat{G}} - \mu_U \frac{U}{G} \frac{\hat{U}}{\hat{G}}$$

Step 10I. Multiply both sides by -1 :

$$\tau_G^* = \mu_G + (\mu_B - \tau_B) \frac{B}{G} \frac{\hat{B}}{\hat{G}} + \mu_U \frac{U}{G} \frac{\hat{U}}{\hat{G}}$$

Step 11I. Now treat $\hat{\tau}_G$ as the exogenous shock. Rewrite equation (4.4) as $\hat{\tau}_B = -\frac{E_{TG}}{E_{TB}} \hat{\tau}_G$ and substitute into equations (4.1)–(4.3) to get three equations analogous to (4.5–4.7) in the text:

$$\hat{G} = E_{GG} \hat{\tau}_G \text{ where } E_{GG} \equiv \varepsilon_{GG} - \varepsilon_{GB} \frac{E_{TG}}{E_{TB}} \quad (4.5')$$

$$\hat{B} = E_{BG} \hat{\tau}_G \text{ where } E_{BG} \equiv \varepsilon_{BG} - \varepsilon_{BB} \frac{E_{TG}}{E_{TB}} \quad (4.6')$$

$$\hat{U} = E_{UG} \hat{\tau}_G \text{ where } E_{UG} \equiv \varepsilon_{UG} - \varepsilon_{UB} \frac{E_{TG}}{E_{TB}}, \quad (4.7')$$

Then use equations (4.5')–(4.7') for the GE changes ($\hat{G}$, $\hat{B}$, and $\hat{U}$) into the equation for τ_G^* above, and cancel the shock $\hat{\tau}_G$ from the numerator and denominator of each ratio to arrive at a closed-form solution for the best constrained τ_G^* (given $\tau_U = 0$ and a fixed τ_B that might not be optimal):

$$\tau_G^* = \mu_G + \underbrace{(\mu_B - \tau_B) \frac{B}{G} \left(\frac{E_{BG}}{E_{GG}} \right)}_{\text{Revenue Term}} + \underbrace{\mu_U \frac{U}{G} \left(\frac{E_{UG}}{E_{GG}} \right)}_{\text{Corlett-Hague Term}}$$

Appendix J – Solving for Both SBO Tax Rates Simultaneously

The equations above for τ_B^* and τ_G^* do not yet determine the levels of the two SBO tax rates that could operate simultaneously to maximize utility (given the untaxable nature of ugly). Each equation gives the best constrained value of one tax rate, for an existing value of the other tax rate that is arbitrary (and could be non-optimal). Equations for τ_B^* and τ_G^* in Appendix I may appear to be two linear equations in two unknowns, but they do not consider the required government revenue. Steps 1J to 8J prove that “best constrained” equations in Appendix I are co-linear. Steps 9J to 18J use the revenue requirement in equation (3.17) as the second independent

equation to solve for simultaneous SBO tax rates.

Proof: the “best constrained” tax rate equations in Appendix I are co-linear.

Step 1J. Use the equation for τ_G^* above to replace τ_G in the equation for τ_B^* to get:

$$\tau_B^* = \mu_B + \left(\mu_G - \left[\mu_G + (\mu_B - \tau_B) \frac{B}{G} \left(\frac{E_{BG}}{E_{GG}} \right) + \mu_U \frac{U}{G} \left(\frac{E_{UG}}{E_{GG}} \right) \right] \right) \frac{G}{B} \left(\frac{E_{GB}}{E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 2J. Cancel the μ_G terms and distribute the minus sign outside the square bracket to get:

$$\tau_B^* = \mu_B + \left[-(\mu_B - \tau_B) \frac{B}{G} \left(\frac{E_{BG}}{E_{GG}} \right) - \mu_U \frac{U}{G} \left(\frac{E_{UG}}{E_{GG}} \right) \right] \frac{G}{B} \left(\frac{E_{GB}}{E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 3J. Bring $\frac{G}{B}$ inside the square bracket so that $\frac{B}{G} \frac{G}{B}$ cancels in the first term and $\frac{U}{G} \frac{G}{B}$ simplifies to $\frac{U}{B}$ in the second term, to get:

$$\tau_B^* = \mu_B + \left[-(\mu_B - \tau_B) \left(\frac{E_{BG}}{E_{GG}} \right) - \mu_U \frac{U}{B} \left(\frac{E_{UG}}{E_{GG}} \right) \right] \left(\frac{E_{GB}}{E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 4J. Distribute $\frac{E_{GB}}{E_{BB}}$ across the square bracket to get:

$$\tau_B^* = \mu_B + \left[-(\mu_B - \tau_B) \left(\frac{E_{BG} E_{GB}}{E_{GG} E_{BB}} \right) - \mu_U \frac{U}{B} \left(\frac{E_{UG} E_{GB}}{E_{GG} E_{BB}} \right) \right] + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 5J. Expand $(\mu_B - \tau_B) \left(\frac{E_{BG} E_{GB}}{E_{GG} E_{BB}} \right)$ and remove the square bracket to get:

$$\tau_B^* = \mu_B - \mu_B \frac{E_{BG} E_{GB}}{E_{GG} E_{BB}} + \tau_B \left(\frac{E_{BG} E_{GB}}{E_{GG} E_{BB}} \right) - \mu_U \frac{U}{B} \left(\frac{E_{UG} E_{GB}}{E_{GG} E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 6J. In equations (4.5)–(4.7) and (4.5')–(4.7'), E_{ij} is the full GE elasticity for the percentage change in disposal i from a one percent change in the tax on disposal j . The mathematical

definitions of these GE elasticities are $E_{iB} \equiv \varepsilon_{iB} - \varepsilon_{iG} \frac{E_{TB}}{E_{TG}}$ and $E_{iG} \equiv \varepsilon_{iG} - \varepsilon_{iB} \frac{E_{TG}}{E_{TB}}$. Notice that

the symmetry between these two definitions is due to the balanced budget constraint in equation (4.4), where revenue raised from an increase in the tax on disposal j requires an offsetting

change in the other tax rate. Therefore we can rewrite each E_{iG} as $-\frac{E_{TG}}{E_{TB}} E_{iB}$ because $E_{iG} \equiv \varepsilon_{iG} -$

$\varepsilon_{iB} \frac{E_{TG}}{E_{TB}} = -\frac{E_{TG}}{E_{TB}} \left[\varepsilon_{iB} - \varepsilon_{iG} \frac{E_{TB}}{E_{TG}} \right] = -\frac{E_{TG}}{E_{TB}} E_{iB}$. Substitute $-\frac{E_{TG}}{E_{TB}} E_{iB}$ for each E_{iG} in the resulting

equation from Step 5J to get:

$$\tau_B^* = \mu_B - \mu_B \frac{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{BB}}{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{BB}} + \tau_B \left(\frac{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{BB}}{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{BB}} \right) - \mu_U \frac{U}{B} \left(\frac{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{UB}}{-\frac{E_{TG}}{E_{TB}} E_{GB} E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 7J. Cancel the common factors from the numerator and denominator of each ratio to get:

$$\tau_B^* = \mu_B - \mu_B + \tau_B - \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right) + \mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)$$

Step 8J. Cancel the two μ_B terms and the two μ_U terms to get:

$$\tau_B^* = \tau_B$$

The substitution of τ_G^* in Step 1J yields a best constrained rate equal to its initial equilibrium rate. The “best constrained” equations in Appendix I describe the same exact linear equation. Nothing in the two co-linear equations determines the two tax rates that together collect the required government revenue, so we use the revenue requirement in equation (3.17) as the second independent equation.

Solve for the SBO Tax on the Bad:

Step 9J. Equation (3.17) says $T = \tau_G G + \tau_B B$. The revenue constraint also applies at the optimum rates, so $T = \tau_G^* G + \tau_B^* B$. Subtract $\tau_B^* B$ from both sides, and divide by G to get:

$$\tau_G^* = \frac{T - \tau_B^* B}{G}$$

Step 10J. Substitute this expression for τ_G^* into the equation for τ_B^* in Appendix I to get:

$$\tau_B^* = \mu_B + \left(\mu_G - \frac{T - \tau_B^* B}{G} \right) \frac{G}{B} \frac{E_{GB}}{E_{BB}} + \mu_U \frac{U}{B} \frac{E_{UB}}{E_{BB}}$$

Step 11J. Expand $\frac{T - \tau_B^* B}{G}$ and cancel $\frac{G}{B}$ with $\frac{B}{G}$ in the τ_B^* term to get:

$$\tau_B^* = \mu_B + \left(\mu_G - \frac{T}{G} \right) \frac{G}{B} \frac{E_{GB}}{E_{BB}} + \tau_B^* \frac{E_{GB}}{E_{BB}} + \mu_U \frac{U}{B} \frac{E_{UB}}{E_{BB}}$$

Step 12J. Subtract $\tau_B^* \frac{E_{GB}}{E_{BB}}$ from both sides, factor out τ_B^* on the left side, and rearrange to get:

$$\tau_B^* \left(1 - \frac{E_{GB}}{E_{BB}} \right) = \mu_B + \mu_U \frac{U}{B} \frac{E_{UB}}{E_{BB}} + \left(\mu_G - \frac{T}{G} \right) \frac{G}{B} \frac{E_{GB}}{E_{BB}}$$

Step 13J. Divide both sides by $\left(1 - \frac{E_{GB}}{E_{BB}} \right)$ to arrive at equation (5.1) in the text:

$$\tau_B^* = \frac{\mu_B + \underbrace{\mu_U \frac{U}{B} \left(\frac{E_{UB}}{E_{BB}} \right)}_{\text{Corlett-Hague Term}} + \underbrace{\left(\mu_G - \frac{T}{G} \right) \frac{G}{B} \left(\frac{E_{GB}}{E_{BB}} \right)}_{\text{Revenue Term}}}{1 - E_{GB}/E_{BB}} \quad (5.1)$$

Solve for the SBO Tax on the Good:

Step 14J. Return to Step 9J, where $T = \tau_G^* G + \tau_B^* B$. Subtract $\tau_G^* G$ from both sides, and then

divide by B to get:

$$\tau_B^* = \frac{T - \tau_G^* G}{B}$$

Step 15J. Substitute this expression for τ_B^* into the equation for τ_G^* in Appendix I to get:

$$\tau_G^* = \mu_G + \left(\mu_B - \frac{T - \tau_G^* G}{B} \right) \frac{B}{G} \frac{E_{BG}}{E_{GG}} + \mu_U \frac{U}{G} \frac{E_{UG}}{E_{GG}}$$

Step 16J. Expand $\frac{T - \tau_G^* G}{B}$ and simplify to get:

$$\tau_G^* = \mu_G + \left(\mu_B - \frac{T}{B} \right) \frac{B}{G} \frac{E_{BG}}{E_{GG}} + \tau_G^* \frac{E_{BG}}{E_{GG}} + \mu_U \frac{U}{G} \frac{E_{UG}}{E_{GG}}$$

Step 17J. Subtract $\tau_G^* \frac{E_{BG}}{E_{GG}}$ from both sides, factor out τ_G^* on the left side, and rearrange to get:

$$\tau_G^* \left(1 - \frac{E_{BG}}{E_{GG}} \right) = \mu_G + \mu_U \frac{U}{G} \frac{E_{UG}}{E_{GG}} + \left(\mu_B - \frac{T}{B} \right) \frac{B}{G} \frac{E_{BG}}{E_{GG}}$$

Step 18J. Divide both sides by $\left(1 - \frac{E_{BG}}{E_{GG}} \right)$ to arrive at equation (5.2) in the text:

$$\tau_G^* = \frac{\mu_G + \underbrace{\mu_U \frac{U}{G} \left(\frac{E_{UG}}{E_{GG}} \right)}_{\text{Corlett-Hague Term}} + \underbrace{\left(\mu_B - \frac{T}{B} \right) \frac{B}{G} \left(\frac{E_{BG}}{E_{GG}} \right)}_{\text{Revenue Term}}}{1 - E_{BG}/E_{GG}} \quad (5.2)$$

Appendix K – Calibration

Here, we describe our hypothetical calibration, later used to illustrate optimal policies. In Table 2, the first column reports assumed allocations of primary factors, in shares θ_{Ki} that sum to one, as required by text equation (3.8). We assume $\theta_{KX} = \theta_{KY} = 0.35$, and $\theta_{KG} = \theta_{KB} = \theta_{KU} = 0.1$. The tax is assumed to be ten percent of the consumer's price of disposal, for each disposal method (so $\theta_{G\tau} = \theta_{B\tau} = 0.1$). Thus, by equations (3.9) and (3.10), we are assuming that the private cost shares are $\theta_{Gp} = \theta_{Bp} = 0.9$. Lastly, in this first column, the elasticity of substitution between X and Y in utility, σ_ψ , is assumed to be one.

The middle column reports implied values of other parameters. If we normalize income to 1.0 and all pre-tax prices to 1.0, then $\theta_{KG} = \theta_{KB} = \theta_{KU} = 0.1$ in the first column implies that disposal quantities are $G = B = U = 0.1$. Then the implied breakdown of total waste into three types of disposal must sum to one, as in equation (3.14), so $\theta_{XG} = \theta_{XB} = \theta_{XU} = 0.333$. Full spending on good X includes the production cost of 0.35 and three disposal costs that add to

0.30. Those costs sum to 0.65, so equation (3.13) implies that the production cost share must be $\theta_{pX} = \frac{0.35}{0.65} = 0.538$ and the disposal cost share must be $\theta_{pW} = \frac{0.30}{0.65} = 0.462$.

Next, the full cost of good disposal is the sum of private cost ($p_K c_G G$, which is 0.9 of the full cost $p_G G$) plus tax cost ($\tau_G G$, which is 0.1 of the full cost). Thus, we divide the normalized private cost per unit of disposal ($p_K c_G = 1.0$) by the private share (0.9) to get the cum-tax price of disposal ($p_G = 1.111$). By our symmetrical assumptions, the initial p_B is also 1.111. Then, multiply each full price by the tax share (0.1) to get each tax rate ($\tau_G = \tau_B = 0.111$). Both taxed disposal methods have the same tax rate and quantity, so each provides half of the revenue ($\theta_{TG} = \theta_{TB} = 0.5$). Plug those values into equation (3.17) to get tax revenue $T = \tau_G G + \tau_B B = (0.111)(0.1) + (0.111)(0.1) = 0.0222$ (as a fraction of national income).

The rightmost column reports marginal external damage from each disposal method and the assumed Allen elasticities. Chosen damage parameter values satisfy $0 < \mu_G < \mu_B < \mu_U$, namely: $\mu_G = 0.05$, $\mu_B = 0.15$, and $\mu_U = 0.5$. We also assume that every cross-price Allen elasticity is $e_{ij} = 0.5$. The three disposal shares are equal, so the adding-up constraint of the Allen elasticities implies that each own-price elasticity is $e_{ii} = -1$.

Table 3 maps the Allen PE elasticities into intermediate and full GE elasticities. Each intermediate elasticity is the percent change in the quantity of one type of disposal from a one percent change in one tax rate, before the revenue constraint requires an offsetting change in the other tax rate. Substitute values from Table 2 into the definitions for the intermediate elasticities in equations (4.1)-(4.3). Symmetry across assumed parameter values makes all intermediate cross-price elasticities equal to:

$$\varepsilon_{ij} = [0.5(0.9) - (-1)(0.1) - 0.5(0.1) - (1)(0.462)(0.35)](0.333)(0.1) = 0.011.$$

By the adding-up constraint ($e_{iG}\theta_{XG} + e_{iB}\theta_{XB} + e_{iU}\theta_{XU} = 0$, for $i = G, B, U$), the own-price intermediate elasticities are equal to:

$$\varepsilon_{ii} = [0.5(-0.1) + (-1)(0.9) - 0.5(0.1) - (1)(0.462)(0.35)](0.333)(0.1) = -0.039.$$

Then, in the rightmost column, Table 3 reports the GE elasticities for our hypothetical initial equilibrium. Plug in values for the intermediate elasticities and the shares of tax revenue from good and bad disposal methods into the definitions following equation (4.4) to get the GE elasticities of total tax revenue:

$$E_{TG} = E_{TB} = 0.5 + 0.5(-0.039) + 0.5(0.011) = 0.486.$$

Then use the definitions in equations (4.5)-(4.7) to calculate the full GE elasticities E_{ij} . Due to the symmetry of our assumptions about disposal in the initial equilibrium, the initial $E_{TB}/E_{TG} = 1$. In this case, equations (4.5)-(4.7) imply that each full GE elasticity in our initial equilibrium is simply a difference between intermediate elasticities:

$$E_{UB} = \varepsilon_{UB} - \varepsilon_{UG} = 0, \quad E_{GB} = \varepsilon_{GB} - \varepsilon_{GG} = 0.050, \quad \text{and} \quad E_{BB} = \varepsilon_{BB} - \varepsilon_{BG} = -0.050.$$

Next, Table 4 shows how a relatively small change in just one of the initial cross-price Allen elasticities can flip the sign of a GE elasticity. The first column retains earlier assumptions for Allen *cross*-price elasticities (0.5), except that e_{UB} is increased to 2.0. Simultaneously, we adjust e_{UU} and e_{BB} to satisfy the adding-up constraint for the Allen elasticities (described in Appendix D and here in Appendix K above), so the Allen elasticities e_{UU} and e_{BB} are both -2.5 .

The intermediate elasticities are calculated in the manner described for Table 3 above. Values for ε_{UB} and ε_{BB} are altered to reflect new initial Allen elasticities in the first column. The other intermediate elasticities are unchanged because ε_{UG} , ε_{BG} , and ε_{GG} do not involve e_{UB} or e_{BB} , and the changes offset each other in ε_{GB} . The new GE elasticities also reflect the change in e_{UB} and the simultaneous change in e_{BB} . Values for the rightmost column are calculated as described for Table 3.

Appendix L – Replication Code for Figure 2

```
In [1]: import numpy as np
import matplotlib.pyplot as plt
from matplotlib.lines import Line2D
```

```
In [2]: # Parameter values from Table 2
BASE = dict(
    theta_KX=0.35, theta_KY=0.35, theta_KG=0.10, theta_KB=0.10, theta_KU=0.10,
    theta_XG=1/3, theta_XB=1/3, theta_XU=1/3,
    theta_pX=0.35/0.65, theta_pW=0.30/0.65,
    theta_Gtau=0.10, theta_Btau=0.10,
    theta_TG=0.5, theta_TB=0.5,
    sigma=1.0, T=(0.10/0.90)*0.1 + (0.10/0.90)*0.1,
    mu_G=0.05, mu_B=0.15, mu_U=0.5,
    G_X=0.1, B_X=0.1, U_X=0.1,
    e_UG=0.5, e_UB=0.5, e_GB=0.5, e_BG=0.5,
    e_GG=-1.0, e_BB=-1.0, e_UU=-1.0)
BASE['e_BG'] = BASE['e_GB']
```

```
In [3]: # Allen adding-up constraint
def adding_up_own(eUG, eUB, eGB, eBG, tXG, tXB, tXU):
    return -(eGB*tXB + eUG*tXU)/tXG,
           -(eBG*tXG + eUB*tXU)/tXB,
           -(eUG*tXG + eUB*tXB)/tXU
```

```
In [4]: # SBO tax rates from equations (5.1) and (5.2)
def compute_sbo(p):
    tKY, tKG, tKB, tKU = p['theta_KY'], p['theta_KG'], p['theta_KB'], p['theta_KU']
    tXG, tXB = p['theta_XG'], p['theta_XB']
    tpW = p['theta_pW']
    tGt, tBt = p['theta_Gtau'], p['theta_Btau']
    tTG, tTB = p['theta_TG'], p['theta_TB']
    s, T, mG, mB, mU = p['sigma'], p['T'], p['mu_G'], p['mu_B'], p['mu_U']
    GX, BX, UX = p['G_X'], p['B_X'], p['U_X']
    eUG, eUB, eGB, eBG = p['e_UG'], p['e_UB'], p['e_GB'], p['e_BG']
    eGG, eBB = p['e_GG'], p['e_BB']
    st = s * tpW * tKY
    # intermediate elasticities, equations (4.1)-(4.3)
    eps_UB = (eUB*(1-tKU) - eBB*tKB - eGB*tKG - st) * tXB * tBt
    eps_UG = (eUG*(1-tKU) - eGG*tKG - eBG*tKB - st) * tXG * tGt
    eps_BB = (eUB*(-tKU) + eBB*(1-tKB) - eGB*tKG - st) * tXB * tBt
    eps_BG = (eUG*(-tKU) - eGG*tKG + eBG*(1-tKB) - st) * tXG * tGt
    eps_GB = (eUB*(-tKU) - eBB*tKB + eGB*(1-tKG) - st) * tXB * tBt
    eps_GG = (eUG*(-tKU) + eGG*(1-tKG) - eBG*tKB - st) * tXG * tGt
    # revenue GE elasticities, equation (4.4)
    E_TG = tTG + tTG*eps_GG + tTB*eps_BG
    E_TB = tTB + tTB*eps_BB + tTG*eps_GB
    if abs(E_TG) < 1e-12 or abs(E_TB) < 1e-12: return np.nan, np.nan
    # full GE elasticities, equations (4.5)-(4.7)
    rB = E_TB / E_TG
    E_GB = eps_GB - eps_GG*rB; E_BB = eps_BB - eps_BG*rB; E_UB = eps_UB - eps_UG*rB
    rG = E_TG / E_TB
    E_GG = eps_GG - eps_GB*rG; E_BG = eps_BG - eps_BB*rG; E_UG = eps_UG - eps_UB*rG
    if abs(E_BB) < 1e-12 or abs(E_GG) < 1e-12: return np.nan, np.nan
    # Simultaneous SBO tax rates, equations (5.1) and (5.2)
    den_B = 1 - E_GB/E_BB
    den_G = 1 - E_BG/E_GG
    if abs(den_B) < 1e-12 or abs(den_G) < 1e-12: return np.nan, np.nan
    tau_B = (mB + (mG - T/GX)*(GX/BX)*(E_GB/E_BB) + mU*(UX/BX)*(E_UB/E_BB)) / den_B
    tau_G = (mG + (mB - T/BX)*(BX/GX)*(E_BG/E_GG) + mU*(UX/GX)*(E_UG/E_GG)) / den_G
    return tau_B, tau_G
```

```
In [5]: # Confirming Allen Properties
def allen_ok(p):
    """Own-price elasticities negative and the weighted Allen matrix
    negative semidefinite (cost minimization requires both)."""
    if p['e_GG'] >= 0 or p['e_BB'] >= 0 or p['e_UU'] >= 0: return False
    th = np.array([p['theta_XG'], p['theta_XB'], p['theta_XU']])
    e = np.array([[p['e_GG'], p['e_GB'], p['e_UG']],
                  [p['e_GB'], p['e_BB'], p['e_UB']],
                  [p['e_UG'], p['e_UB'], p['e_UU']]])
    return np.linalg.eigvalsh(th[:, None] * e * th[None, :]).max() <= 1e-9
```

```
In [6]: # Varying Parameters
def set_eUB(p, v):
    """Change e_UB; adjust own-price elasticities to satisfy adding-up."""
    p['e_UB'] = v
    p['e_GG'], p['e_BB'], p['e_UU'] = adding_up_own(
        p['e_UG'], p['e_UB'], p['e_GB'], p['e_BG'],
        p['theta_XG'], p['theta_XB'], p['theta_XU'])
    return allen_ok(p)

def set_thetaKU(p, v):
    """Change theta_KU; rescale the other theta_K's from their primary values
    to sum to one and recompute the expenditure shares (apply to a fresh copy
    of BASE; modifiers are not meant to be chained)."""
    if v <= 0.01 or v >= 0.99: return False
    sectors = ['theta_KX', 'theta_KY', 'theta_KG', 'theta_KB']
    old_sum = sum(BASE[s] for s in sectors)
```

```

p['theta_KU'] = v
for s in sectors:
    p[s] = BASE[s] * (1.0 - v) / old_sum
cost_G = p['theta_KG'] / (1 - p['theta_Gtau'])
cost_B = p['theta_KB'] / (1 - p['theta_Btau'])
cost_U = p['theta_KU']
pW_X = cost_G + cost_B + cost_U
pF_X = p['theta_KX'] + pW_X
p['theta_pX'] = p['theta_KX'] / pF_X; p['theta_pW'] = pW_X / pF_X
p['theta_XG'] = cost_G / pW_X; p['theta_XB'] = cost_B / pW_X; p['theta_XU'] =
cost_U / pW_X
p['e_GG'], p['e_BB'], p['e_UU'] = adding_up_own(
    p['e_UG'], p['e_UB'], p['e_GB'], p['e_BG'],
    p['theta_XG'], p['theta_XB'], p['theta_XU'])
return allen_ok(p)

```

In [7]:

```

# Check the primary calibration against values in Tables 2 and 3
tb, tg = compute_sbo(dict(BASE))
print(f"Primary calibration: tau_B* = {tb:.6f}, tau_G* = {tg:.6f}")
print(f"Revenue check: {tg*BASE['G_X'] + tb*BASE['B_X']:.6f} = T = {BASE['T']:.6f}")
p = dict(BASE); st = p['sigma']*p['theta_pW']*p['theta_KY']
eps_UB = (p['e_UB']*(1-p['theta_KU']) - p['e_BB']*p['theta_KB'] -
p['e_GB']*p['theta_KG'] - st) \
    * p['theta_XB']*p['theta_Btau']
eps_BB = (p['e_UB']*(-p['theta_KU']) + p['e_BB']*(1-p['theta_KB']) -
p['e_GB']*p['theta_KG'] - st) \
    * p['theta_XB']*p['theta_Btau']
print(f"Table 2 check: eps_UB = {eps_UB:.3f} (0.011), eps_BB = {eps_BB:.3f} (-0.039)")

```

Primary calibration: tau_B* = 0.161111, tau_G* = 0.061111
Revenue check: 0.022222 = T = 0.022222
Table 2 check: eps_UB = 0.011 (0.011), eps_BB = -0.039 (-0.039)

In [8]:

```

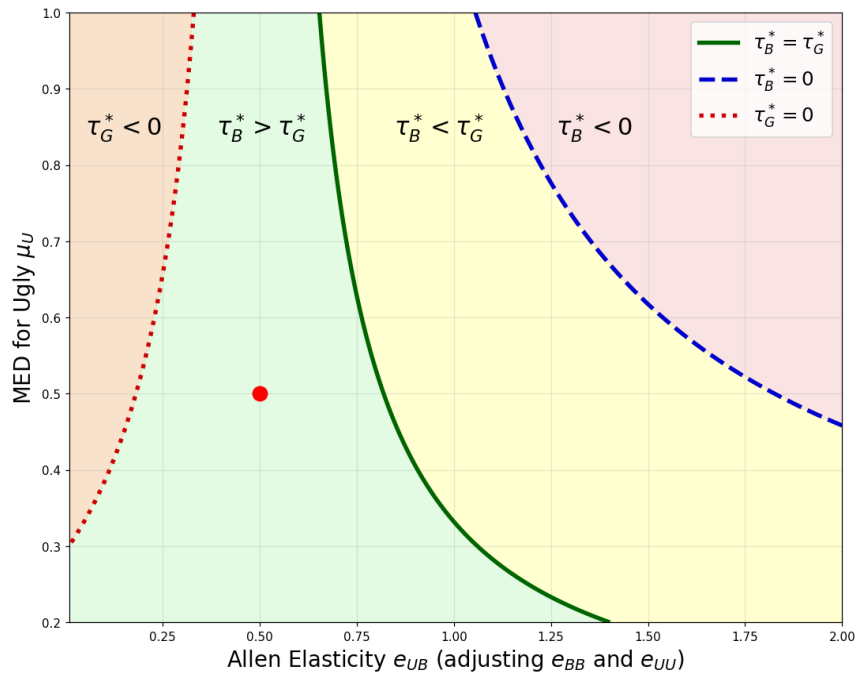
# Four-region plot
def plot_regions(x, y, tauB, tauG, xlabel, ylabel, dot, fname, labels=()):
    X, Y = np.meshgrid(x, y)
    diff = tauB - tauG
    fig, ax = plt.subplots(figsize=(10, 8))
    for tx, ty, s in labels:
        ax.text(tx, ty, s, fontsize=20, fontweight='bold', transform=ax.transAxes)
    ax.contourf(X, Y, ((diff > 0) & (tauB >= 0) & (tauG >= 0)).astype(float),
        levels=[0.5, 1.5], colors=['#c8f7c8'], alpha=0.5)
    ax.contourf(X, Y, ((diff < 0) & (tauB >= 0) & (tauG >= 0)).astype(float),
        levels=[0.5, 1.5], colors=['#ffffb0'], alpha=0.6)
    ax.contourf(X, Y, (tauB < 0).astype(float), levels=[0.5, 1.5], colors=['#f7c8c8'],
        alpha=0.5)
    ax.contourf(X, Y, (tauG < 0).astype(float), levels=[0.5, 1.5], colors=['#f5d0a9'],
        alpha=0.6)
    ax.contour(X, Y, diff, levels=[0], colors='#006400', linewidths=3.5)
    ax.contour(X, Y, tauB, levels=[0], colors='#0000cc', linewidths=3.5, linestyle='--')
    ax.contour(X, Y, tauG, levels=[0], colors='#cc0000', linewidths=3.5,
        linestyle=':')
    ax.plot(*dot, 'o', color='red', markersize=12, zorder=10)
    ax.legend(handles=[
        Line2D([0], [0], color='#006400', lw=3.5, label=r'$\tau^*_B = \tau^*_G$'),
        Line2D([0], [0], color='#0000cc', lw=3.5, ls='--', label=r'$\tau^*_B = 0$'),
        Line2D([0], [0], color='#cc0000', lw=3.5, ls=':', label=r'$\tau^*_G = 0$')],
        loc='upper right', fontsize=16)
    ax.set_xlabel(xlabel, fontsize=18); ax.set_ylabel(ylabel, fontsize=18)
    ax.grid(True, alpha=0.3)
    plt.tight_layout(); plt.savefig(fname, dpi=150, bbox_inches='tight')
n = 200

```

In [9]:

```
# Panel A:  $e_{UB}$  vs.  $\mu_U$ 
x = np.linspace(0.01, 2.0, n); y = np.linspace(0.2, 1.0, n)
tauB = np.full((n, n), np.nan); tauG = np.full((n, n), np.nan)
for i, mu_U in enumerate(y):
    for j, e_UB in enumerate(x):
        p = dict(BASE); p['mu_U'] = mu_U
        if not set_eUB(p, e_UB): continue
        tauB[i, j], tauG[i, j] = compute_sbo(p)
plot_regions(x, y, tauB, tauG,
             r'Allen Elasticity  $e_{UB}$  (adjusting  $e_{BB}$  and  $e_{UU}$ )',
             r'MED for Ugly  $\mu_U$ ', (0.5, 0.5), 'panelA.png',
             labels=[(0.02, 0.80, r'\tau^*_G < 0'), (0.19, 0.80, r'\tau^*_B > \tau^*_G'),
                    (0.42, 0.80, r'\tau^*_B < \tau^*_G'), (0.63, 0.80, r'\tau^*_B < 0')],
             < 0$'))

def crossing(fun, lo, hi):
    flo = fun(lo)
    for _ in range(60):
        mid = 0.5*(lo + hi)
        if np.sign(fun(mid)) == np.sign(flo): lo = mid
        else: hi = mid
    return 0.5*(lo + hi)
def at(e_UB, k):
    p = dict(BASE)
    if not set_eUB(p, e_UB): return np.nan
    tB, tG = compute_sbo(p)
    return {'tG': tG, 'diff': tB - tG, 'tB': tB}[k]
zG = crossing(lambda v: at(v, 'tG'), 0.05, 0.5)
zBG = crossing(lambda v: at(v, 'diff'), 0.5, 1.5)
zB = crossing(lambda v: at(v, 'tB'), 1.0, 2.0)
print(f"Panel A at  $\mu_U = 0.5$ :  $\tau_G^* = 0$  at  $e_{UB} = \{zG:.4f\}$ ; "
      f" $\tau_B^* = \tau_G^*$  at  $\{zBG:.4f\}$ ;  $\tau_B^* = 0$  at  $\{zB:.4f\}$ ")
```



In [10]:

```
# Panel B:  $\theta_{KU}(x)$  vs  $\mu_B(y)$  with  $e_{UB} = 2$ 
x2 = np.linspace(0.01, 0.25, n); y2 = np.linspace(0.0501, 0.50, n)
tauB2 = np.full((n, n), np.nan); tauG2 = np.full((n, n), np.nan)
```

```

for i, mu_B in enumerate(y2):
    for j, tKU in enumerate(x2):
        p = dict(BASE); p['e_UB'] = 2.0
        if not set_thetaKU(p, tKU): continue
        # keep mu_B > mu_G
        if mu_B <= p['mu_G']: continue
        p['mu_B'] = mu_B
        tauB2[i, j], tauG2[i, j] = compute_sbo(p)
plot_regions(x2, y2, tauB2, tauG2,
             r"Factor Share for Ugly  $\theta_{KU}$  (other  $\theta_{Ki}$ 's rescaled and  $e_{UB}=2$ )",
             r'MED for Bad  $\mu_B$ ', (0.10, 0.15), 'panelB.png',
             labels=[(0.60, 0.90, r' $\tau^*_G < 0$ '), (0.60, 0.63, r' $\tau^*_B > \tau^*_G$ '),
                    (0.60, 0.30, r' $\tau^*_B < \tau^*_G$ '), (0.60, 0.05, r' $\tau^*_B < 0$ ')]
pB = dict(BASE); pB['e_UB'] = 2.0; set_thetaKU(pB, 0.10)
tb2, tg2 = compute_sbo(pB)
print(f"Panel B red dot ( $\theta_{KU}=0.10$ ,  $\mu_B=0.15$ ,  $e_{UB}=2$ ):  $\tau_B^* = \{tb2:.4f\}$ ,  $\tau_G^* = \{tg2:.4f\}$ ")

```

